Question (Andrew Paul): Given a constant gravitational acceleration and initial velocity, and assuming that the ground is flat, will a projectile be displaced the same amount horizontally from the time it is launched to the time it hits the ground if it is launched twice and the second launch angle was the complement of the first?

Solution (Andrew Paul): First, we generate and graph various parabolic trajectories keeping the constants constant and only taking the complements of angles. The trajectories are generated from the equation derived earlier in *Making the Soccer Problem*:

$$y_{\vec{s}} = \frac{\vec{a}_g x_{\vec{s}}^2}{2v_i^2 \cos^2\theta} + x_{\vec{s}} \tan\theta$$

Figure 2.1 seems to suggest that the conjecture is true. But how do we prove it? Notice that the equation provided above designates the starting point as the origin and the landing point as the second root of the parabola. Hence, the second root of the parabola is also equivalent to the horizontal displacement of the projectile. Therefore, we must see if the equation for the parabolic trajectory has the same second root as the equation that has the same constants except the angle which is the complement.

First, we factor the trajectory equation in order to find its roots in terms of the constants. We get:

$$x_{\vec{s}}\left(\frac{\vec{a}_g x_{\vec{s}}}{2v_i^2 \cos^2\theta} + \tan\theta\right)$$

From this factored form, we find the roots to be 0 (which is the starting point) and:

$$x_{\vec{s}} = -\frac{2v_i^2 \cos^2\theta \tan\theta}{\vec{a}_g} = -\frac{v_i^2 \sin 2\theta}{\vec{a}_g}$$

which is also the horizontal displacement. Ergo it suffices to show that:

$$-\frac{v_i^2 \sin 2\theta}{\vec{a}_g} = -\frac{v_i^2 \sin\left[2(90^\circ - \theta)\right]}{\vec{a}_g} \Rightarrow \sin 2\theta = \sin\left[2(90^\circ - \theta)\right]$$

This is trivial as we have:

$$\sin\left[2(90^\circ - \theta)\right] = 2\sin(90^\circ - \theta)\cos(90^\circ - \theta) = 2\cos\theta\sin\theta = \sin 2\theta$$

Hence we can conclude that under the same gravity with the same initial velocity on a flat ground, launching a projectile at complementary angles will result in equal horizontal displacements.

Sidenote: Figure 2.1 suggests that $\theta = 45^\circ$ maximizes the horizontal displacement. This is true as obviously $-\frac{v_i^2 \sin 2\theta}{\vec{a}_g}$ is maximized when $\theta = 45^\circ$.

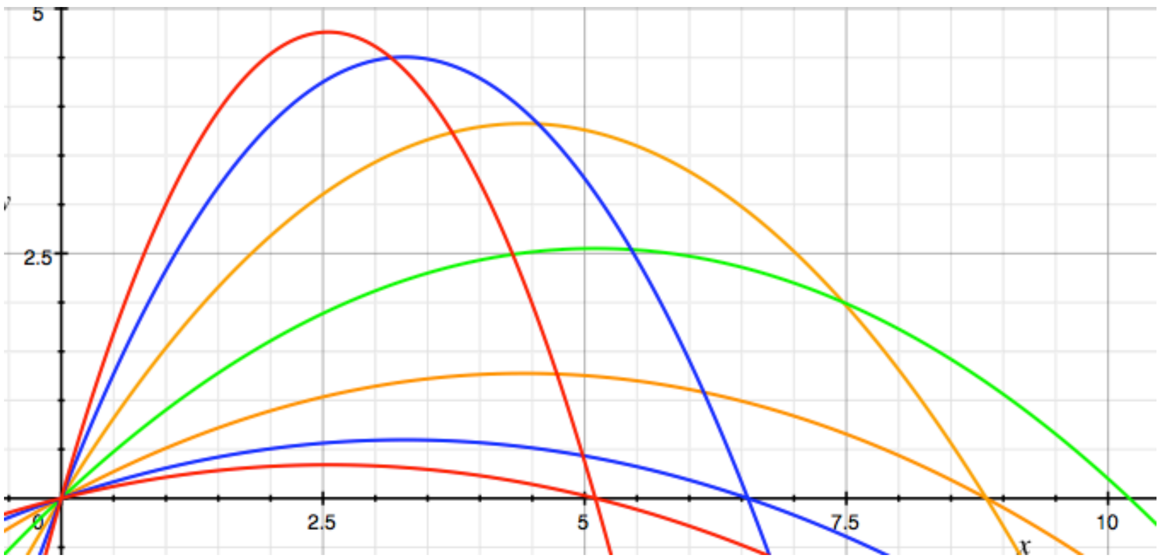

Fig 2.1: Each color is an angle-complement pair. The trajectories were generated assuming an initial velocity of 10 m/s and a gravitational acceleration of –9.8 m/s^2. Red is 75 and 15 degrees, blue is 70 and 20 degrees, orange is 30 and 60 degrees, and green is 45 degrees.