A Second Order Linear ODE

Andrew Paul

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Consider the homogeneous ODE

$$\ddot{x} - Kx = 0.$$

How do we solve this equation? Observe that

$$\begin{aligned}\ddot{x} &= \frac{\mathrm{d}\dot{x}}{\mathrm{d}t} \\ &= \frac{\mathrm{d}\dot{x}}{\mathrm{d}x} \cdot \frac{\mathrm{d}x}{\mathrm{d}t} \\ &= \dot{x}\frac{\mathrm{d}\dot{x}}{\mathrm{d}x}\end{aligned}$$

Hence, we can perform a separation,

$$\int \dot{x} \ \mathrm{d}\dot{x} = \int Kx \ \mathrm{d}x,$$

which becomes

$$\dot{x} = \sqrt{Kx^2 + C_1}.$$

We chose the principal square root, WLOG. It becomes evident that the choice of root does not change the general solution. We may separate again to obtain,

$$I = \int \frac{\mathrm{d}x}{\sqrt{Kx^2 + C_1}} = t + C_2.$$

Now let us compute the integral I. Let $x = \sqrt{\frac{C_1}{K}} \tan\theta$. Then, $\mathrm{d}x = \sqrt{\frac{C_1}{K}} \sec^2\theta \ \mathrm{d}\theta$, and the integral becomes

$$\begin{aligned}I &= \frac{1}{\sqrt{K}} \int \sec\theta \ \mathrm{d}\theta \\ &= \frac{1}{\sqrt{K}} \log\left(\sec\theta + \tan\theta\right) \\ &= \frac{1}{\sqrt{K}} \log\left(x\sqrt{\frac{K}{C_1}} + \sqrt{1 + \frac{K}{C_1}x^2}\right).\end{aligned}$$

Hence, we have

$$\frac{1}{\sqrt{K}}\log\left(x\sqrt{\frac{K}{C_1}}+\sqrt{1+\frac{K}{C_1}x^2}\right)=t+C_2.$$

After a lot of algebraic manipulation, we obtain

$$x=\frac{e^{C_2\sqrt{K}}}{2K}e^{t\sqrt{K}}-\frac{C_1}{2e^{C_2\sqrt{K}}}e^{-t\sqrt{K}}.$$

Now if $K>0$, we are done. The solution is simply,

$$\boxed{x=Ae^{t\sqrt{K}}+Be^{-t\sqrt{K}}}$$

for $A,B\in\mathbb{R}$. However, if $K<0$, as is the case in simple harmonic motion as given by Hooke's law, we need to make a few more observations.

First of all, the number $e^{C_2\sqrt{K}}$ becomes nonreal. Letting $\omega=\sqrt{|K|}$, it becomes the complex number $z=e^{i\omega C_2}$. The number $e^{-C_2\sqrt{K}}$ is then the conjugate, $\overline{z}$. Hence the solution to the differential equation can be written as

$$\begin{aligned}x&=\frac{e^{C_2\sqrt{K}}}{2K}e^{t\sqrt{K}}-\frac{C_1}{2e^{C_2\sqrt{K}}}e^{-t\sqrt{K}}\\&=\frac{z}{2K}e^{i\omega t}-\frac{C_1\overline{z}}{2}e^{-i\omega t}\\&=Ae^{t\sqrt{K}}+Be^{-t\sqrt{K}}\\&=Ae^{i\omega t}+Be^{-i\omega t},\end{aligned}$$

where this time $A,B\in\mathbb{C}$. What are the ramifications of this? We apply Euler's formula, $e^{i\theta}=\cos\theta+i\sin\theta$ to expand the solution:

$$Ae^{i\omega t}+Be^{-i\omega t}=A\cos\omega t+Ai\sin\omega t+B\cos\omega t-Bi\sin\omega t=(A+B)\cos\omega t+(A-B)i\sin\omega t.$$

Note that we also applied the even and odd properties of cosine and sine, respectively.

Since we are seeking real-valued solutions, we need a way to resolve all imaginary terms in our solution such that they vanish. Suppose that $A=p+qi$ and $B=r+si$. For $A+B\in\mathbb{R}$, we must have $q+s=0$ or $q=-s$. Furthermore, for $(A-B)i\in\mathbb{R}$, we must have $p-r=0$ or $p=r$. Therefore, for a real solution, we must have $A=\overline{B}$.

We have previously defined $A=\frac{z}{2K}$ and $B=\frac{C_1\overline{z}}{2}$. Since these must be conjugates, we can obtain:

$$\frac{C_1\overline{z}}{2}=\overline{\left(\frac{z}{2K}\right)}\Rightarrow C_1=\frac{1}{K}.$$

This is an interesting condition. Only a particular constant of integration following our first separation of variables allows for a real-valued solution when $K<0$. We conclude that the solution for

$K < 0$ is of the form:

$$\boxed{x = Ae^{t\sqrt{K}} + \overline{A}e^{-t\sqrt{K}}}$$

for $A \in \mathbb{C}$. Our argument to show that $A = \overline{B}$ also paves the way to show that this solution is in fact sinusoidal. We can express the solution as:

$$\boxed{x = C\cos\omega t + D\sin\omega t}$$

where $C, D \in \mathbb{R}$. Condensing this into a single trigonometric function is trivial. It is simply

$$\boxed{\sqrt{C^2 + D^2}\sin\left(\omega t + \phi\right)},$$

where $\phi = \arcsin\frac{C}{\sqrt{C^2+D^2}}$.

This is why motion under Hooke's law, simple harmonic motion, is sinusoidal.