Putnam 2019 Problems

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Here are my solutions to the two problems that I managed to solve on the Putnam.

2019 A1: Determine all possible values of

$$A^3 + B^3 + C^3 - 3ABC$$

where A, B, and C are nonnegative integers.

Solution: Let us denote the expression by f. First observe that by AM-GM,

$$\frac{A^3 + B^3 + C^3}{3} \geq \sqrt[3]{A^3B^3C^3} \Rightarrow f \geq 0,$$

so f must be nonnegative. $f = 0$ is obtained trivially by letting $A = B = C = 0$.

Now, let WLOG, $A = B = n$ and $C = n - 1$ for some positive integer n, then,

$$\begin{aligned} f &= 2n^3 + (n-1)^3 - 3n^2(n-1) \\ &= 2n^3 + n^3 - 3n^2 + 3n - 1 - 3n^3 + 3n^2 \\ &= 3n - 1, \end{aligned}$$

so f can be any positive integer congruent to 2 modulo 3. Furthermore, setting WLOG $A = B = n - 1$ and $C = n$, we obtain $f = 3n - 2$. So f can be 0 and any positive integer not divisible by 3.

Suppose that we have an f that is divisible by 3 Then, we must have,

$$f = A^3 + B^3 + C^3 - 3ABC \equiv A^3 + B^3 + C^3 \equiv A + B + C \equiv 0 \pmod 3,$$

where in the last step, we apply the fact that for any integer n, we have $n^3 \equiv n \pmod 3$. This can be shown by looking at all three possible residues modulo 3.

There are precisely four cases where $A + B + C \equiv 0 \pmod 3$.

Case 1: $(A, B.C) \equiv (0, 0, 0) \pmod 3$

In this case, since all three variables have at least one factor of 3, their cubes must have at least three factors of 3. Furthermore, the term $3ABC$ must have at least four factors of 3. Hence, we must have

$$f \equiv 0 \pmod 9.$$

Case 2: $(A, B, C) \equiv (1, 1, 1) \pmod 3$

In this case, we can write $A = 3x + 1$, $B = 3y + 1$, and $C = 3z + 1$ for $x, y, z \in \mathbb{Z}$. Expanding the expression for f, we see that every coefficient in the polynomial is divisible by 9. Hence, in this case

$$f \equiv 0 \pmod 9.$$

Case 3: $(A, B, C) \equiv (2, 2, 2) \pmod 3$

Similarly to the previous case, we let $A = 3x + 2$, $B = 3y + 2$, and $C = 3z + 2$ and see that every coefficient in the expansion of f is divisible by 9, hence

$$f \equiv 0 \pmod 9.$$

Case 4: $(A, B, C) \equiv$ some permutation of $(0, 1, 2) \pmod 9$

In this case, WLOG let the permutation in consideration be $(0, 1, 2)$. Then A^3 is divisible by 9 by the argument made in the first case, and likewise $3ABC$ is divisible by 9 because it has at least two factors of 3. Now we can write the remaining terms as

$$B^3 + C^3 = (B + C)(B^2 + C^2 - BC).$$

But $B + C \equiv 1 + 2 \equiv 0 \pmod 3$ and $B^2 + C^2 - BC \equiv 1 + 4 - 2 \equiv 0 \pmod 3$. So $B^3 + C^3$ can be factored into two integers which are themselves divisible by 3, hence $B^3 + C^3 \equiv 0 \pmod 9$. Hence, once again, in this case we have

$$f \equiv 0 \pmod 9.$$

So we have shown that whenever f is a multiple of 3, it is also a multiple of 9. Now, we simply let WLOG $A = n$, $B = n - 1$, and $C = n - 2$ for some integer n larger than 1. From this, we obtain $f = 9(n - 1)$, so we can construct every positive multiple of 9.

So f can be all nonnegative integers except for multiples of 3 which are not multiples of 9. $\square$

2019 B1: Denote by $\mathbb{Z}^2$ the set of all points (x, y) in the plane with integer coordinates. For each integer $n \geq 0$, let P_n be the subset of $\mathbb{Z}^2$ consisting of the point $(0, 0)$ together with all points (x, y) such that $x^2 + y^2 = 2^k$ for some integer $k \leq n$. Determine, as a function of n, the number of four-point subsets of P_n whose elements are the vertices of a square.

Solution: We denote the circle whose radius is $\frac{k}{2}$ as Ω_k. Observe that for $k < 0$, Ω_k is bounded by the unit circle, Ω_0, and there exists no lattice points in this region apart from the origin itself, so no square will have a vertex on $\Omega_k \forall k < 0$.

On the complex plane, the vertices of every square whose vertices are all on a single circle, Ω_k, can be written as

$$\{2^{\frac{k}{2}}e^{i\theta}, 2^{\frac{k}{2}}e^{i(\theta+\frac{\pi}{2})}, 2^{\frac{k}{2}}e^{i(\theta+\pi)}, 2^{\frac{k}{2}}e^{i(\theta+\frac{3\pi}{2})}\},$$

since the central angle of a square is $\frac{\pi}{2}$.

Using Euler's formula and the angle addition identities for sine and cosine, we can compute the real and imaginary parts of each of these complex numbers. Let the above complex numbers be $\{z_1, z_2, z_3, z_4\}$. Then,

$$\Re(z_1) = 2^{\frac{k}{2}} \cos\theta$$

$$\Im(z_1) = 2^{\frac{k}{2}} \sin\theta$$

$$\Re(z_2) = 2^{\frac{k}{2}} \cos\left(\theta + \frac{\pi}{2}\right) = -2^{\frac{k}{2}} \sin\theta$$

$$\Im(z_2) = 2^{\frac{k}{2}} \sin\left(\theta + \frac{\pi}{2}\right) = 2^{\frac{k}{2}} \cos\theta$$

$$\Re(z_3) = 2^{\frac{k}{2}} \cos(\theta + \pi) = -2^{\frac{k}{2}} \cos\theta$$

$$\Im(z_3) = 2^{\frac{k}{2}} \sin(\theta + \pi) = -2^{\frac{k}{2}} \sin\theta$$

$$\Re(z_4) = 2^{\frac{k}{2}} \cos\left(\theta + \frac{3\pi}{2}\right) = 2^{\frac{k}{2}} \sin\theta$$

$$\Im(z_4) = 2^{\frac{k}{2}} \sin\left(\theta + \frac{3\pi}{2}\right) = -2^{\frac{k}{2}} \cos\theta.$$

So if z_1 is a Gaussian integer, then the other three complex numbers are as well. WLOG, let $0 \leq \theta \leq \frac{\pi}{2}$.

For z_1 to be a Gaussian integer, we must have $2^{\frac{k}{2}} \cos\theta \in \mathbb{Z}$.

Case 1: k is even

Since the product of a rational number and a irrational number is irrational (given that both numbers are nonzero), we must have that $\cos\theta$ is rational. Suppose that $\cos\theta = \frac{p}{q}$, where $\frac{p}{q}$ is irreducible. For this real part to be an integer, q must be a divisor of $2^{\frac{k}{2}}$, as otherwise it would leave some product of prime numbers in the denominator, and since $\gcd(p, q) = 1$ (if p is nonzero), we cannot remove these prime numbers by p. The divisors of $2^{\frac{k}{2}}$ are all of the powers of 2 from 1 to $2^{\frac{k}{2}}$. So,

$$q \in \{1, 2, ..., 2^{\frac{k}{2}}\}.$$

Now, we must also have that $2^{\frac{k}{2}} \sin\theta$, the imaginary part, is an integer. This is

$$\begin{aligned}\Im(z_1) &= 2^{\frac{k}{2}} \sin\theta \\ &= 2^{\frac{k}{2}} \sqrt{1-\cos^2\theta} \\ &= 2^{\frac{k}{2}} \frac{\sqrt{q^2-p^2}}{q}.\end{aligned}$$

Since q must be a factor of $2^{\frac{k}{2}}$, $\frac{2^{\frac{k}{2}}}{q}$ is an integer power of 2. Hence, we must have $q^2 - p^2 = \frac{\ell^2}{2^j}$, where the fraction is fully reduced, for some $\ell, j \in \mathbb{Z}$. In particular, we have that $0 \le \frac{j}{2} \le \frac{k}{2} - \log_2 q$, as otherwise, we divide away too many powers of 2 from $2^{\frac{k}{2}}$ and end up with a non-integer. Furthermore, j must be even so that $\sqrt{q^2-p^2}$ is rational.

The equation rearranges to

$$2^j q^2 - 2^j p^2 = \ell^2.$$

When j is positive, observe that the LHS is congruent to 0 modulo 4. But if ℓ^2 is congruent to 0 modulo 4, ℓ must be even, which contradicts the irreducibility of $\frac{\ell^2}{2^j}$, unless $\ell = 0$, in which case $p = q$ so $\cos\theta = 1$. We would then obtain the Gaussian integers at angles 0, $\frac{\pi}{2}$, π, and $\frac{3\pi}{2}$, which form a square.

When $j = 0$, we have $q^2 - p^2 = \ell^2$. Then, we note that since q is an power of 2, we can write this as

$$4^s - p^2 = \ell^2$$

for some nonnegative integer s. If $s = 0$, then we must have $p = 1$ or $p = 0$ so that the irreducibility of $\cos\theta = \frac{p}{q}$ is not contradicted. Regardless of which of these two values we choose for p, we obtain the same four complex numbers that we found in the previous case, namely those with arguments of 0, $\frac{\pi}{2}$, π, and $\frac{3\pi}{2}$. These form a square.

If $s > 0$, we have that $-p^2 \equiv \ell^2 \pmod 4$. Since perfect squares are only congruent to 0 or 1 modulo 4, we must have that $p^2 \equiv 0 \pmod 4$, so p is even. The only even p that does not break the irreducibility condition of $\frac{p}{q}$ is $p = 0$, which as before, yields the square with vertices at the complex numbers with arguments 0, $\frac{\pi}{2}$, π, and $\frac{3\pi}{2}$.

Hence, we can only have $p = 0$ so $\cos\theta = 0$. This means, when k is even, there exists only one square whose vertices are all on Ω_k, namely at angles 0, $\frac{\pi}{2}$, π, and $\frac{3\pi}{2}$.

Case 2: k is odd

In this case, the radius of Ω_k is $2^{\frac{k-1}{2}}\sqrt{2}$. So, for the real part of z_1 to be an integer, we must have $\cos\theta = \frac{1}{\sqrt{2}} \cdot \frac{p}{q}$ where $2^{\frac{k-1}{2}} \cdot \frac{p}{q}$ is an integer, and $\frac{p}{q}$ is irreducible. Similarly to the even case, we must have

$$q \in \{1, 2, ..., 2^{\frac{k-1}{2}}\}.$$

Now, the imaginary part of z_1 is

$$\begin{aligned}\Im(z_1) &= 2^{\frac{k}{2}}\sin\theta \\ &= 2^{\frac{k}{2}}\sqrt{1-\cos^2\theta} \\ &= 2^{\frac{k-1}{2}}\frac{\sqrt{2q^2-p^2}}{q}.\end{aligned}$$

Once again, we can write $2q^2 - p^2 = \frac{\ell^2}{2^j}$, for a nonnegative even j and an integer ℓ such that $\frac{\ell^2}{2^j}$ is irreducible.

When $j > 0$, we have

$$2^{j+1}q^2 - 2^j p^2 = \ell^2,$$

The LHS is congruent to 0 modulo 4 but ℓ^2 is congruent to 0 modulo 4, ℓ must be even, which contradicts the irreducibility of $\frac{\ell^2}{2^j}$ unless $\ell = 0$. If $\ell = 0$, we obtain $\cos\theta = 1$, but this does not yield a Gaussian integer since $2^{\frac{k}{2}}$ is not an integer for odd k.

When $j = 0$, we have $2q^2 - p^2 = \ell^2$. Since $q = 2^s$ for some nonnegative integer s, we have

$$2^{2s+1} - p^2 = \ell^2.$$

When $s = 0$, we have $p^2 + \ell^2 = 2$. Since we have $q = 1$, the only possible values of p are 0 or 1, since any other values of p would cause $\cos\theta$ to exceed 1. We find that $p = q = 1$ yields a valid perfect square $\ell^2 = 1$. When $p = q$, we have $\cos\theta = \sin\theta = \frac{1}{\sqrt{2}}$. So we have the complex numbers on Ω_k with arguments $\frac{\pi}{4}$, $\frac{3\pi}{4}$, $\frac{5\pi}{4}$, and $\frac{7\pi}{4}$. This forms a square.

When $s \neq 0$, we have that $-p^2 \equiv \ell^2 \pmod 4$. Since squares are only congruent to 0 or 1 modulo 4, we must have that $p^2 \equiv 0 \pmod 4$ so p is even. The only p that is even that does not contradict the irreducibility of $\frac{p}{q}$ is $p = 0$, but this gives us the same complex numbers we discussed in a previous case. They are not Gaussian integers.

Hence, we can only have $p = q$, so $\cos\theta = \frac{1}{\sqrt{2}}$. This means, when k is odd, there exists only one square whose vertices are all on Ω_k, namely at angles $\frac{\pi}{4}$, $\frac{3\pi}{4}$, $\frac{5\pi}{4}$, and $\frac{7\pi}{4}$.

Due to the arrangement of the lattice points on adjacent circles (they alternate between the even and odd orientations), we can show by geometric arguments that we cannot form a square with two vertices on one circle and two vertices on another.

We can, however, form squares taking the origin, two lattice points on Ω_k and a lattice point on Ω_{k+1} for nonnegative $k < n$. These configurations exist since the radius of Ω_k, satisfies the recurrence

$$R_{k+1} = R_k\sqrt{2}.$$

This is exactly the factor we multiply the side length of a square by to obtain its diagonal. Furthermore, the angles formed between the side length that is the radius on Ω_k and the diagonal which is the radius on Ω_k is exactly $\frac{\pi}{4}$ due to the angles that we have computed previously. The two possible configurations are shown below.

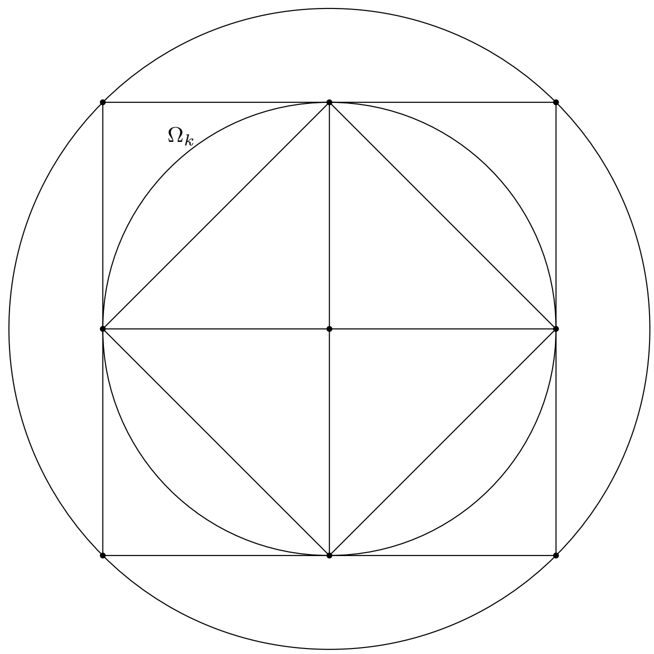

Figure 1: The configuration when k is even

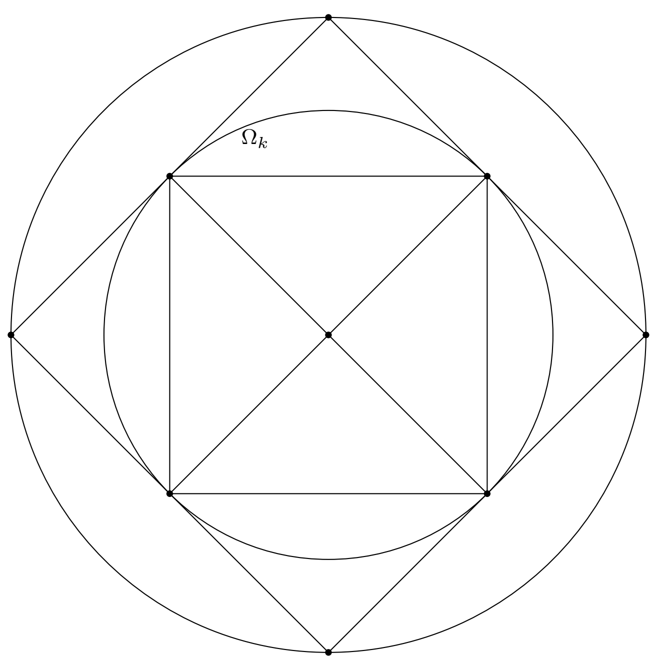

Figure 2: The configuration when k is odd

Hence, each circle Ω_k from $k = 0$ to $k = n - 1$ contributes exactly 5 squares and the circle Ω_n

contributes only one square, since there is no circle beyond it to take points from. So the answer is $\boxed{5n+1}$.