Chapter 1

Statics

3. Motionless chain *

A frictionless surface is in the shape of a function which has its endpoints at the same height but is otherwise arbitrary. A chain of uniform mass per unit length rests on this surface, from end to end. Show that the chain will not move.

Solution: Suppose that the function is $f(x)$, over the interval $[a, b]$. We are given $f(a) = f(b)$. Let the chain have a linear mass density, $\frac{\mathrm{d}m}{\mathrm{d}\ell} = \lambda$. Then, the force an infinitesimally small section of chain experiences down the slope is:

$$\mathrm{d}F = \lambda g \sin\theta \ \mathrm{d}\ell,$$

where θ is the angle of inclination the tangent to f makes at a particular x. Observe that:

$$\mathrm{d}\ell = \sqrt{(\mathrm{d}x)^2 + (\mathrm{d}f)^2} = \sqrt{1 + f'(x)^2} \ \mathrm{d}x.$$

Furthermore, $\theta = \arctan f'(x)$ so $\sin\theta = \frac{f'(x)}{\sqrt{1+f'(x)^2}}$. Hence, we have:

$$\mathrm{d}F = \lambda g f'(x) \ \mathrm{d}x,$$

so the net force on the rope is given by:

$$F = \int \mathrm{d}F = \lambda g \int_a^b f'(x) \ \mathrm{d}x = \lambda g[f(b) - f(a)] = 0,$$

as desired. □

4. Keeping the book up

A book of mass M is positioned against a vertical wall. The coefficient of friction between the book and the wall is μ. You wish to keep the book from falling by pushing on it with a force F applied at an angle *theta* to the horizontal $(-\pi/2 < \theta < \pi/2)$. For a given θ, what is the minimum F required? What is the limiting value θ for which there exists an F which will keep the book up?

Solution: The normal force on the book is $F\cos\theta$, hence the force of friction on the book is $\mu F \cos\theta$. The force upwards that you exert on the book is $F\sin\theta$. These two forces must cancel weight in equilibrium. Hence, the least force F that keeps the system in equilibrium satisfies:

$$\mu F\cos\theta + F\sin\theta = mg,$$

or:

$$F = \frac{mg}{\sin\theta + \mu\cos\theta}.$$

Such an F may become arbitrarily large. It becomes infinite when:

$$\sin\theta + \mu\cos\theta = 0 \Rightarrow \theta = -\arctan\mu.$$

□

5. Objects between circles **

Each of the following planar objects is placed, as shown in Fig. 1, between two frictionless circles of radius R. The mass density of each object is σ, and the radii to the points of contact make an angle θ with the horizontal. For each case, find the horizontal force that must be applied to the circles to keep them together. For what θ is this force maximum or minimum?

(a) An isosceles triangle with common side length L.

(b) A rectangle with height L.

(c) A circle.

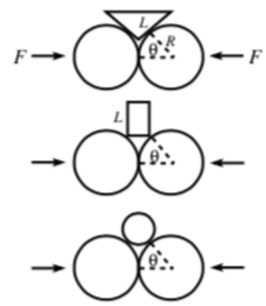

Figure 1.1: Problem 5

Solution: This is simply a geometry problem.

(a) Using geometry, we find that the vertex angle of the isosceles triangle must be 2θ. Hence, the area of the triangle is $\frac{1}{2}L^2\sin 2\theta$ and the weight of the triangle is $\frac{1}{2}\sigma g L^2 \sin 2\theta$. Each circle exerts a vertical force of $F = N\cos\theta$ on the triangle, where N is the normal force, hence:

$$2N\sin\theta = \frac{1}{2}\sigma g L^2 \sin 2\theta.$$

This reduces to:

$$\boxed{F = \frac{1}{2}\sigma g L^2 \cos^2\theta},$$

which is 0 for $\theta = \frac{\pi}{2}$ and increases as θ decreases. □

(b) Geometry reveals that the width of the rectangle must be $2R(1-\cos\theta)$. Therefore, the weight of the rectangle is $2\sigma g R L(1-\cos\theta)$. Therefore:

$$2N\sin\theta = 2\sigma g R L(1-\cos\theta).$$

This yields:

$$\boxed{F = \sigma g R L(1-\cos\theta)\cot\theta}.$$

The derivative of this function is:

$$F' = \sigma g R L\left(\sin\theta\cot\theta - (1-\cos\theta)\csc^2\theta\right).$$

Setting this equal to 0, we find:

$$\cos\theta = \frac{1-\cos\theta}{1-\cos^2\theta},$$

or:

$$\cos^3\theta - 2\cos\theta + 1 = 0,$$

which factors as:

$$(\cos\theta - 1)(\cos^2\theta + \cos\theta - 1) = 0.$$

The quadratic factor yields:

$$\cos\theta = \frac{-1\pm\sqrt{5}}{2}.$$

And due to the range of cosine, the only viable solution is $\theta = \arccos\frac{-1+\sqrt{5}}{2}$. This is where the force required is maximized. Incidentally, this angle is equal to $\arccos\varphi$ where $\varphi = \frac{-1+\sqrt{5}}{2}$ is the golden ratio. $\square$

(c) Applying the Law of Sines on the triangle whose vertices are the centers of the three circles, we find that the radius of the third circle is $r = R(\sec\theta - 1)$. Hence, the weight of that circle is $\sigma g\pi R^2(\sec\theta-1)^2$. Therefore:

$$2N\sin\theta = \sigma g\pi R^2(\sec\theta - 1)^2,$$

and:

$$\boxed{F = \frac{1}{2}\sigma g\pi R^2(\sec\theta-1)^2\cot\theta}.$$

L'hopital's rule shows that $\lim_{\theta\to 0} F = 0$. F also diverges as $\theta \to \frac{\pi}{2}$. $\square$

16. Leaning sticks and circles **

A large number of sticks (of mass per unit length ρ) and circles (of radius R) lean on each other, as shown in Fig. 2. Each stick makes an angle θ with the ground. Each stick is tangent to a circle at its upper end. The sticks are hinged to the ground, and every other surface is *frictionless*. In the limit of a very large number of sticks and circles, what is the normal force between a stick and the circle it rests on, very far to the right? (Assume that the last circle is glued to the floor, to keep it from moving.)

Figure 1.2: Problem 16

Solution: Consider the n^{th} stick. Let the force that the n^{th} stick exerts on the n^{th} circle be F_n. Let the force that the n^{th} stick exerts on the $(n-1)^{\text{th}}$ be G_n (with obviously $G_1 = 0$). The torque that F_n exerts on the n^{th} stick about the hinge is thus simply $F_n\ell$ where ℓ is the length of the stick, since the stick is tangential to the circle.

The moment arm of the torque by G_n is more complicated. Let it be A. Then, the torque on the n^{th} stick about the hinge by G_n is AG_n.

The moment arm of the torque by gravity (weight) is simpler. Since gravity is vertical, the angle it makes with the stick is $90^\circ - \theta$. Since ρ is constant, the center of mass of the stick is a distance of $\frac{\ell}{2}$ from the hinge. The moment arm of the torque by gravity is thus $\frac{1}{2}\ell\cos\theta$. It follows that the torque due to gravity on every stick is $\frac{1}{2}\rho g\ell^2\cos\theta$.

Balancing the torques on the n^{th} stick:

$$\frac{1}{2}\rho g\ell^2\cos\theta + AG_n - F_n\ell = 0.$$

Now we turn to geometry. We label the configuration as follows.

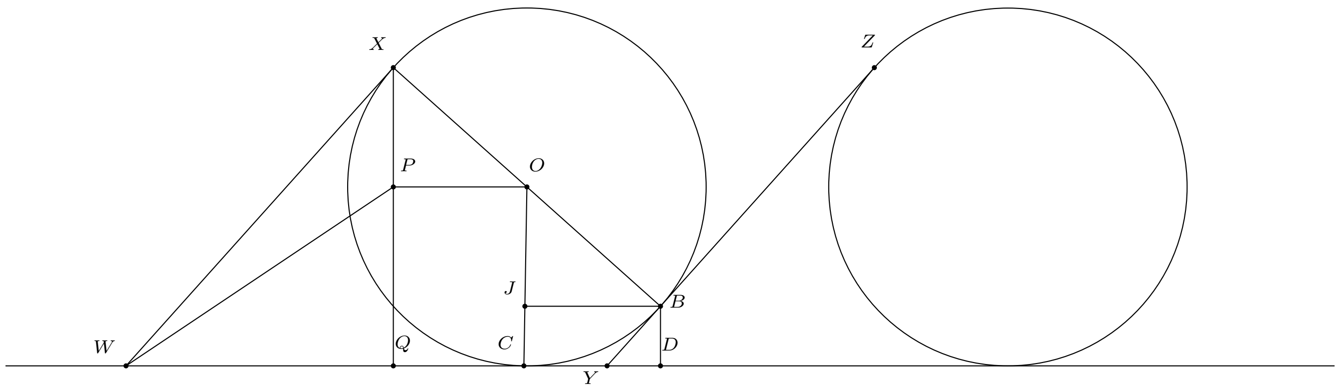

Figure 1.3: Problem 16 Diagram

Let $\overline{YZ}$ be the n^{th} stick and $\overline{WX}$ be the $(n-1)^{\text{th}}$ stick. Let O be the center of the $(n-1)^{\text{th}}$ circle. Suppose $\overline{YZ}$ is tangential to the circle centered at O at B. Let the foot of the altitude from B to the ground be D. Let the point of tangency between the ground and the circle centered at O be C. Let J be foot of the altitude from B to $\overline{OC}$. Let Q be the foot of the altitude from X to the ground. Let P be the foot of the altitude from O to $\overline{OX}$.

Since $\angle OXW = \angle OCW = 90°$, we must have $\theta + \angle COX = 180°$. Since $\angle OCY = \angle OBY = 90°$, we must have $\angle BOC + 180° - \theta = 180°$, so $\angle BOC = \theta$. Now we have $\angle COX + \angle BOC = 180°$, so X, O, and B are collinear. It follows that $\angle POX = 90° - \theta$ and so $\angle OXP = \theta$.

For the $(n-1)^{\text{th}}$ circle to remain in equilibrium, all lateral forces must cancel. Hence $F_{n-1}\sin\theta = G_n \sin\theta$. This yields:

$$F_{n-1} = G_n.$$

Next, we compute ℓ. We have $XP = R\cos\theta$, $PQ = OC = R$, $CW = WX = \ell$ and $QC = OP = R\sin\theta$. Therefore, by the Pythagorean theorem on $\triangle WXQ$:

$$\begin{aligned}\ell^2 &= (\ell - R\sin\theta)^2 + (R + R\cos\theta)^2 \\ &= \ell^2 - 2R\ell\sin\theta + R^2\sin^2\theta + R^2 + 2R^2\cos\theta + R^2\cos^2\theta \\ &= \ell^2 - 2R\ell\sin\theta + 2R^2\cos\theta + 2R^2\end{aligned}$$

This yields:

$$\ell = \frac{R(1+\cos\theta)}{\sin\theta}.$$

Now we compute A, the moment arm of G_n. In particular, $A = BY$. Observe that $BD = JC = R - R\cos\theta$. Hence, by looking at $\triangle BDY$, we obtain:

$$A = BY = \frac{R(1-\cos\theta)}{\sin\theta},$$

therefore, our very first equation, balancing torques, becomes:

$$\frac{1}{2}\rho g\left[\frac{R^2(1+\cos\theta)^2}{\sin^2\theta}\right]\cos\theta + F_{n-1}\left[\frac{R(1-\cos\theta)}{\sin\theta}\right] - F_n\left[\frac{R(1+\cos\theta)}{\sin\theta}\right] = 0.$$

Now suppose $L = \lim_{n\to\infty} F_n$. Then:

$$\frac{1}{2}\rho g\left[\frac{R^2(1+\cos\theta)^2}{\sin^2\theta}\right]\cos\theta + L\left[\frac{R(1-\cos\theta)}{\sin\theta}\right] - L\left[\frac{R(1+\cos\theta)}{\sin\theta}\right] = 0,$$

which simplifies to:

$$\boxed{L = \frac{1}{4}\rho g R\left[\frac{(1+\cos\theta)^2}{\sin\theta}\right]}.$$

17. Balancing the stick **

Given a semi-infinite stick (that is, one that goes off to infinity in one direction), find how its density should depend on position so that it has the following property: if the stick is cut at an arbitrary location, the remaining semi-infinite piece will balance on a support located a distance b from the end.

Solution: Let the stick start at $x = 0$ and extend infinitely in the positive x direction. Let the desired density function be $\lambda(x)$. Suppose the stick is cut at $x = c$. Then the pivot placed a distance b from the end will separate the stick into a finite interval of $[c, b+c)$ and an infinite interval of $(b+c, \infty)$. Since we require the stick to be in equilibrium, the torques about the pivot must cancel:

$$\int_c^{b+c} \lambda(x)(b + c - x)\ \mathrm{d}x = \int_{b+c}^{\infty} \lambda(x)(x - b - c)\ \mathrm{d}x,$$

which rearranges to:

$$\int_c^{\infty} \lambda(x)(b + c - x)\ \mathrm{d}x = 0.$$

This is a *linear Volterra equation of the first kind.* To solve it, we differentiate WRT c. Using the Leibniz integral rule, and assuming that $\lambda(x)$ is sufficiently well-behaved, we may write:

$$\int_c^{\infty} \lambda(x)\ \mathrm{d}x = b\lambda(c).$$

Suppose $M'(x) = \lambda(x)$. Then:

$$\lim_{R\to\infty} M(R) - M(c) = b\lambda(c).$$

Let $L = \lim_{R\to\infty} M(R)$. Then:

$$M'(c) + \frac{1}{b}M(c) = \frac{L}{b}.$$

This is a first-order linear ODE. We apply an integrating factor of $\exp \int \frac{1}{b}\ \mathrm{d}c$. After a change of variables, this yields:

$$M(x) = L + Ce^{-x/b}.$$

Differentiating this, we conclude:

$$\boxed{\lambda(x) = Ce^{-x/b}}.$$

For $C > 0$.

18. The spool **

A spool consists of an axis of radius r and an outside circle of radius R which rolls on the ground. A thread which is wrapped around the axis is pulled with a tension T.

(a) Given R and r, what angle, θ, should the thread make with the horizontal so that the spool does not move. Assume there is large enough friction between the spool and ground so that the spool doesn't slip.

(b) Given R, r, and a coefficient of friction μ between the spool and ground, what is the largest T can be (assuming the spool doesn't move)?

(c) Given R and μ, what should r be so that the upper bound on T found in part (b) is as small as possible (assuming the spool doesn't move)? What is the resulting value of T?

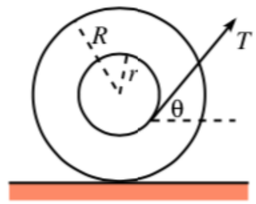

Figure 1.4: Problem 18

Solution: This is similar to Problem 13 in the 2011 $F = ma$.

(a) There must be (static) friction, F_f, at the contact point between the ground and the wheel. For equilibrium to exist, this friction must cancel the horizontal component of tension. Hence, $F_f = T\cos\theta$. The torques must also sum to zero, hence:

$$TR\cos\theta = Tr,$$

hence the desired angle is $\boxed{\theta = \arccos\frac{r}{R}}$.

(b) The normal force that the spool experiences is given by:

$$N = mg - T\sin\theta,$$

where m is the mass of the spool. Hence, the force of friction, which we have shown is $T\cos\theta$, is limited by:

$$T\cos\theta \le \mu(mg - T\sin\theta).$$

This rearranges to:

$$T \le \frac{\mu mg}{\cos\theta + \mu\sin\theta}.$$

Since $\cos\theta = \frac{r}{R}$, we must have $\sin\theta = \frac{1}{R}\sqrt{R^2 - r^2}$, hence:

$$\boxed{T \le \frac{\mu mgR}{r + \mu\sqrt{R^2 - r^2}}}.$$

(c) The denominator of the upper bound can be condensed into a single sinusoidal function with amplitude $\sqrt{1+\mu^2}$. It follows immediately that the least upper bound of T is $\boxed{\frac{\mu mgR}{\sqrt{1+\mu^2}}}$. Condensing the denominator:

$$\cos\theta + \mu\sin\theta = \sqrt{1+\mu^2}\left(\cos\theta \cdot \frac{1}{\sqrt{1+\mu^2}} + \sin\theta \cdot \frac{\mu}{\sqrt{1+\mu^2}}\right).$$

Let $\sin\alpha = \frac{1}{\sqrt{1+\mu^2}}$. Then $\cos\alpha = \frac{\mu}{\sqrt{1+\mu^2}}$. Then the above equation becomes:

$$\cos\theta + \mu\sin\theta = \sin\alpha\cos\theta + \sin\theta\cos\alpha = \sin(\alpha + \theta).$$

We wish for this to equal unity, hence:

$$\alpha + \theta = \frac{\pi}{2},$$

or:

$$\theta = \frac{\pi}{2} - \alpha.$$

Taking the cosine of both sides:

$$\frac{r}{R} = \sin\alpha = \frac{1}{\sqrt{1+\mu^2}}.$$

Hence, we conclude that the r that makes this possible is $\boxed{r = \frac{R}{\sqrt{1+\mu^2}}}$.

Chapter 2

Using $F = ma$

1. Sliding down a plane **

(a) A block slides down a frictionless plane from the point $(0, y)$ to the point $(b, 0)$, where b is given. For what value of y does the journey take the shortest time? What is this time?

(b) Answer the same questions in the case where there is a coefficient of kinetic friction, μ, between the block and the plane.

Solution:

(a) The angle of inclination of the plane is given by $\theta = \arctan \frac{y}{b}$. The acceleration of the block down the plane is given by $g \sin\theta$. The distance it must traverse is given by the Pythagorean theorem, $\sqrt{b^2 + y^2}$. Therefore, by kinematics,

$$\frac{1}{2} g \Delta t^2 \sin\theta = \sqrt{b^2 + y^2} \Rightarrow \Delta t = \sqrt{\frac{2\sqrt{b^2 + y^2}}{g \sin\theta}}.$$

Using Pythagorean identities, we find that $\sin\theta = \frac{y}{\sqrt{b^2+y^2}}$, hence

$$\Delta t = \sqrt{\frac{2(b^2 + y^2)}{gy}}.$$

The square root function is monotonically increasing so it suffices to minimize $\frac{b^2}{y} + y$. By AM-GM,

$$\frac{b^2}{y} + y \geq 2b.$$

And equality occurs when $\frac{b^2}{y} = y$ or $\boxed{y = b}$. It follows that the minimal time is:

$$\min \Delta t = \sqrt{\frac{2}{g} \cdot 2b} = \boxed{2\sqrt{\frac{b}{g}}}.$$

(b) We repeat the same calculation, but note that the acceleration of the block down the plane is now $g \sin\theta - \mu g \cos\theta$ due to friction. Hence,

$$\Delta t = \sqrt{\frac{2\sqrt{b^2 + y^2}}{g(\sin\theta - \mu\cos\theta)}} = \sqrt{\frac{2}{g}\left(\frac{b^2 + y^2}{y - \mu b}\right)},$$

so it suffices to minimize $\frac{b^2+y^2}{y-\mu b}$. Setting the derivative equal to 0:

$$2y(y-\mu b) - b^2 - y^2 = 0 \Rightarrow y^2 - 2\mu b y - b^2 = 0.$$

The roots of this quadratic are $y = b\left(\mu \pm \sqrt{\mu^2+1}\right)$. Since $y > 0$, the minimum must occur at $\boxed{y = b\left(\mu + \sqrt{\mu^2+1}\right)}$. Observe that when $\mu = 0$, this yields $y = b$ as desired. Plugging this in to find the minimal time, we have,

$$\boxed{\min \Delta t = 2\sqrt{\frac{b}{g}\left(\mu + \sqrt{\mu^2+1}\right)}}.$$

Observe that when $\mu = 0$, this reduces to what we found in (a).

Atwood's machine **

(a) A massless pulley hangs from a fixed support. A string connecting two masses, M_1 and M_2, hangs over the pulley. Find the accelerations of the masses.

(b) Consider now the double-pulley system with masses M_1, M_2, and M_3. Find the accelerations of the masses.

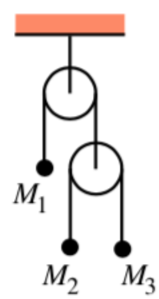

Figure 2.1: The double-pulley system

Solution:

(a) Suppose WLOG that $M_2 \geq M_1$. Since the string is inelastic, the tension is equal at both ends of the string and the accelerations of both masses must be equal in magnitude. That is,

$$\frac{M_2 g - T}{M_2} = \frac{T - M_1 g}{M_1},$$

where T is the tension in the string. This yields $T = \frac{2M_1 M_2 g}{M_1+M_2}$. The acceleration of the masses is then

$$\frac{T - M_1 g}{M_1} = \boxed{\frac{(M_2 - M_1)g}{M_1 + M_2}}.$$

(b) Consider the forces on the second pulley (excluding the masses M_2 and M_3 attached to it). There is one instance of T_1, the tension in first Atwood system, acting upwards, and two instances of T_2, the tension in the second Atwood system, acting downwards. However, observe that the pulley itself is said to be *massless*. By Newton's Second Law the net force on it will always be 0, as otherwise the pulley would have an infinite acceleration since $\lim_{m\to 0} \frac{F}{m} = \infty$ for $F > 0$. In other words, $T_1 - 2T_2 = 0$, or

$$T_2 = \frac{1}{2}T_1.$$

To gain an intuition for this, we consider what happens in the extreme case $M_2 + M_3 \to 0$. In this case, $T_2 = 0$ as otherwise, the string in the second Atwood system would have an infinite acceleration. It follows that $T_1 = 0$. This makes sense physically, as M_1 would simply be in free fall with the string attached to it not held taut by any mass, hence not possessing any tension.

Now we continue our calculations with this in mind. Let a_1 be the acceleration of the first Atwood system and let a_2 be the acceleration of the second Atwood system. WLOG, let $M_3 \geq M_2$. Then,

$$\begin{cases} T - M_1 g = M_1 a_1 \\ \frac{T}{2} - M_2 g = M_2(a_2 - a_1) \\ M_3 g - \frac{T}{2} = M_3(a_1 + a_2) \end{cases}$$

where up is taken to be positive. From this system, we obtain:

$$a_1 = \frac{M_1M_2 + M_1M_3 - 4M_2M_3}{M_1M_2 + M_1M_3 + 4M_2M_3} g,$$

$$a_2 = \frac{2M_1M_2 - 2M_1M_3}{M_1M_2 + M_1M_3 + 4M_2M_3} g.$$

Hence, the acceleration of M_2 is given by:

$$a_2 - a_1 = \boxed{\frac{3\,M_1M_3 - M_1M_2 - 4\,M_2M_3}{M_1M_2 + M_1M_3 + 4\,M_2M_3} g},$$

and the acceleration of M_3 is given by:

$$a_1 + a_2 = \boxed{\frac{M_1M_3 + 4\,M_2M_3 - 3\,M_1M_2}{M_1M_2 + M_1M_3 + 4\,M_2M_3} g}.$$

Maximum length of trajectory ***

A ball is thrown at speed v from zero height on level ground. Let θ_0 be the angle at which the ball should be thrown so that the distance traveled *through the air* is maximum. Show that θ_0 satisfies

$$1 = \sin\theta_0 \log\left(\frac{1 + \sin\theta_0}{\cos\theta_0}\right).$$

(The solution is found numerically to be $\theta_0 \approx 56.5°$).

Solution: Let the ball be thrown at an arbitrary angle θ. Then the parametric equations of motion are

$$\begin{cases} y(t) = vt\sin\theta - \frac{1}{2}gt^2 \\ x(t) = vt\cos\theta. \end{cases}$$

Implicitizing these equations, we find that

$$y = x\tan\theta - \frac{g}{2v^2\cos^2\theta}x^2.$$

The nonzero root of this equation is $\frac{v^2 \sin 2\theta}{g}$, which is the horizontal range of the ball. Therefore, the length of the trajectory is given by

$$\ell = \int_{x_0}^{x_1} \sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\,\mathrm{d}x = \int_0^{\frac{v^2\sin 2\theta}{g}} \sqrt{1 + (A + Bx)^2}\,\mathrm{d}x,$$

where $A = \tan\theta$ and $B = -\frac{g}{v^2\cos^2\theta}$. Let I be the indefinite integral. We perform the trigonometric substitution $A + Bx = \tan\alpha$. This yields

$$I = \frac{1}{B}\int \sec^3\alpha \, \mathrm{d}\alpha.$$

We proceed with integration by parts. Let J be the antiderivative of secant cubed. We let $u = \sec\alpha$, so $\mathrm{d}u = \sec\alpha\tan\alpha \, \mathrm{d}\alpha$ and $\mathrm{d}v = \sec^2\alpha \, \mathrm{d}\alpha$ so $v = \tan\alpha$. This gives us

$$J = \sec\alpha\tan\alpha - \int \sec\alpha\tan^2\alpha \, \mathrm{d}\alpha.$$

But since $\tan^2\alpha = \sec^2\alpha - 1$, we have

$$J = \sec\alpha\tan\alpha - J + \int \sec\alpha \, \mathrm{d}\alpha \Rightarrow J = \frac{1}{2}\left[\sec\alpha\tan\alpha + \log\left(\sec\alpha + \tan\alpha\right)\right].$$

Hence, $I = \frac{1}{2B}\left[\sec\alpha\tan\alpha + \log\left(\sec\alpha + \tan\alpha\right)\right]$. Reversing our substitutions, this is

$$I = \frac{1}{2B}\left[(A+Bx)\sqrt{1+(A+Bx)^2} + \log\left(A + Bx + \sqrt{1+(A+Bx)^2}\right)\right].$$

Using the fundamental theorem of calculus, we obtain

$$\begin{aligned}
\ell &= -\frac{v^2\cos^2\theta}{2g}\left[-\sec\theta\tan\theta + \log\left(\sec\theta - \tan\theta\right) - \sec\theta\tan\theta - \log\left(\sec\theta + \tan\theta\right)\right] \\
&= -\frac{v^2\cos^2\theta}{2g}\left[\log\left(\frac{\sec\theta - \tan\theta}{\sec\theta + \tan\theta}\right) - 2\sec\theta\tan\theta\right] \\
&= -\frac{v^2\cos^2\theta}{2g}\left[\log\left(\frac{2}{1+\sin\theta} - 1\right) - 2\sec\theta\tan\theta\right].
\end{aligned}$$

The maximum value of ℓ occurs at a critical point. This θ_0 is thus the root of the first derivative of ℓ WRT θ. Since we will be setting the first derivative to 0, we can discard the constant $-\frac{v^2}{2g}$. This gives us

$$\begin{aligned}
&-\sin 2\theta_0\left[\log\left(\frac{2}{1+\sin\theta_0} - 1\right) - 2\sec\theta_0\tan\theta_0\right] + \cos^2\theta_0\left(-\frac{2\cos\theta_0}{(1+\sin\theta_0)^2}\cdot\frac{1}{\frac{2}{1+\sin\theta_0} - 1} - 2\sec\theta_0\tan^2\theta_0 - 2\sec^3\theta_0\right) \\
&\Rightarrow \sin\theta_0\log\left(\frac{2}{1+\sin\theta_0} - 1\right) - 2\tan^2\theta_0 + 1 + \tan^2\theta_0 + \sec^2\theta_0 \\
&\Rightarrow \sin\theta_0\log\left(\frac{2}{1+\sin\theta_0} - 1\right) + 2 = 0.
\end{aligned}$$

But, observe that

$$\begin{aligned}
\frac{2}{1+\sin\theta_0} - 1 &= \left(\frac{1+\sin\theta_0}{1-\sin\theta_0}\right)^{-1} \\
&= \left[\frac{(1+\sin\theta_0)(1+\sin\theta_0)}{(1-\sin\theta_0)(1+\sin\theta_0)}\right]^{-1} \\
&= \left(\frac{1+\sin\theta_0}{\cos\theta_0}\right)^{-2}.
\end{aligned}$$

Hence, we have

$$-2\sin\theta_0\log\left(\frac{1+\sin\theta_0}{\cos\theta_0}\right) + 2 = 0 \Rightarrow \boxed{\sin\theta_0\log\left(\frac{1+\sin\theta_0}{\cos\theta_0}\right) = 1},$$

as desired.