Problem (Textbook): Suppose a parabola is defined parametrically as

$$\begin{cases} x(t) = 2at \\ y(t) = at^2 \end{cases}$$

For some constant $a \in \mathbb{R}$. Points P and Q are defined on such a parabola by parameters p and q respectively such that $(0, -a) \in \overleftrightarrow{PQ}$. If F is the focus of the parabola, show that $\frac{1}{FP} + \frac{1}{FQ} = \frac{1}{a}$.

Solution (Andrew Paul): First, we find the equation of $\overleftrightarrow{PQ}$. We are given $P(2ap, ap^2)$ and $Q(2aq, aq^2)$. Thus $m = \frac{ap^2 - aq^2}{2ap - 2aq} = \frac{p+q}{2}$. Hence, the (implicit) equation of the line is $y = \frac{1}{2}(p+q)x + b$. We note that $b = -apq \Rightarrow y = \frac{1}{2}(p+q)x - apq$. We are also given the y-intercept of this line $(0, -a)$. So we have $b = -apq = -a \Rightarrow pq = 1 \Rightarrow p = \frac{1}{q}$.

This means that we now have $Q\left(\frac{2a}{p}, \frac{a}{p^2}\right)$. Now we implicitize the equations for the parabola. This gives us:

$$y = \frac{x^2}{4a}$$

We can see easily now that the focus of this parabola lies at $F(0, a)$. It becomes clear that we can set up the distance formula many times to prove the statement $\frac{1}{FP} + \frac{1}{FQ} = \frac{1}{a}$. However, going just a few steps in, the algebra becomes incredibly complicated and messy.

Instead, let us prove the equivalent statement $a(d_1 + d_2) = d_1 d_2$ where $d_1 = FP$ and $d_2 = FQ$ without using the distance formula. To do so, we recall the definition of a parabola. It is the locus of points that is equidistant from a point (the focus) and a line (the directrix). In the context of our problem, this is a revelation! That is, d_1 and d_2 are not only FP and FQ respectively, but also the shortest distance between P and Q and the directrix respectively!

The directrix of our parabola is seen to be $y = -a$ (simply one focal length from the vertex at the origin but in the opposite direction from the focus). The length from P to this horizontal line is the sum of the absolute value of the y-coordinate of P and a. Thus:

$$d_1 = ap^2 + a$$

Similarly, working with $Q\left(\frac{2a}{p}, \frac{a}{p^2}\right)$, we have:

$$d_2 = \frac{a}{p^2} + a$$

We wish to show $a(d_1 + d_2) = d_1 d_2$, or:

$$a\left(ap^2 + 2a + \frac{a}{p^2}\right) = \left(ap^2 + a\right)\left(\frac{a}{p^2} + a\right)$$

Expanding both sides and simplifying gives us:

$$a^2p^2 + 2a^2 + \frac{a^2}{p^2} = 2a^2 + a^2p^2 + \frac{a^2}{p^2}$$

Which is trivially true for every allowed a, p. **Q.E.D.**

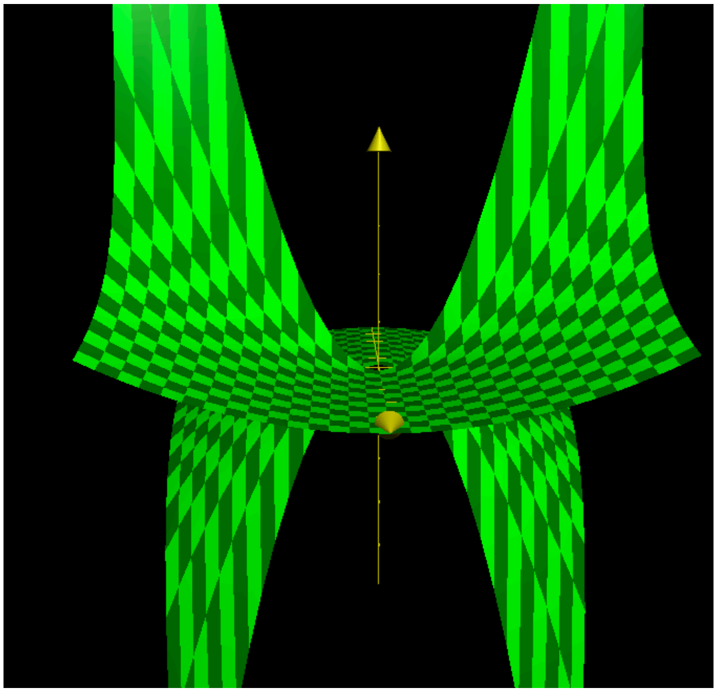

Fig 5.1: The surface in $\mathbb{R}^3$ that describes all possible parabolas given the problem's parametric equations. The axis that would stretch out of the screen is the *a*-axis and cross-sections (by planes parallel to the screen representing constant *a* values) are the parabolas that result.

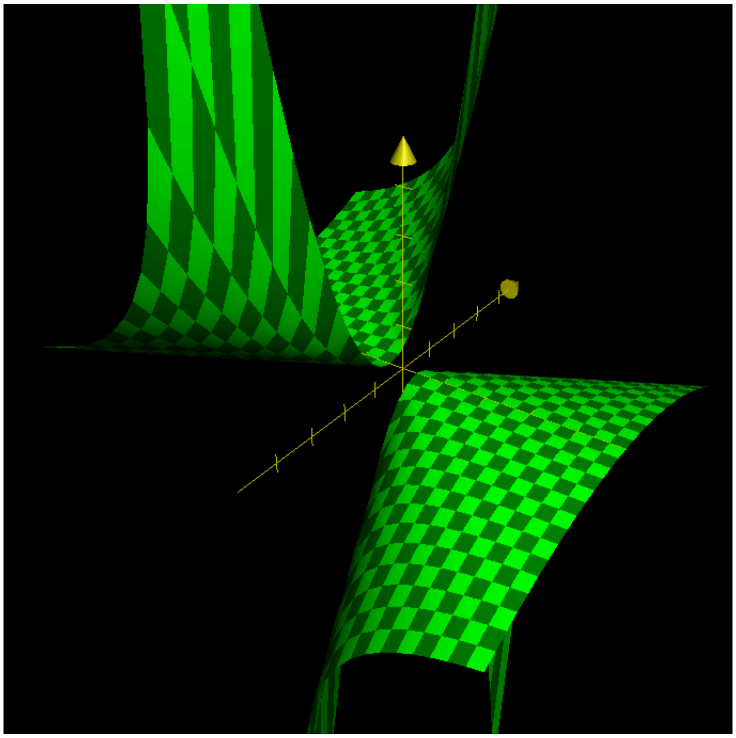

Fig 5.2: Another view of the manifold.