Problem (Math StackExchange): $\triangle ABC$ has a right angle at A with D being the midpoint of $\overline{BC}$. A line through D meets $\overleftrightarrow{AB}$ at X and $\overline{AC}$ at Y. Let M be the midpoint of $\overline{XY}$ and let P be the reflection of D across M. Let the foot of the altitude from P to $\overline{BC}$ be T. Prove that $\overline{AM}$ bisects $\angle TAD$.

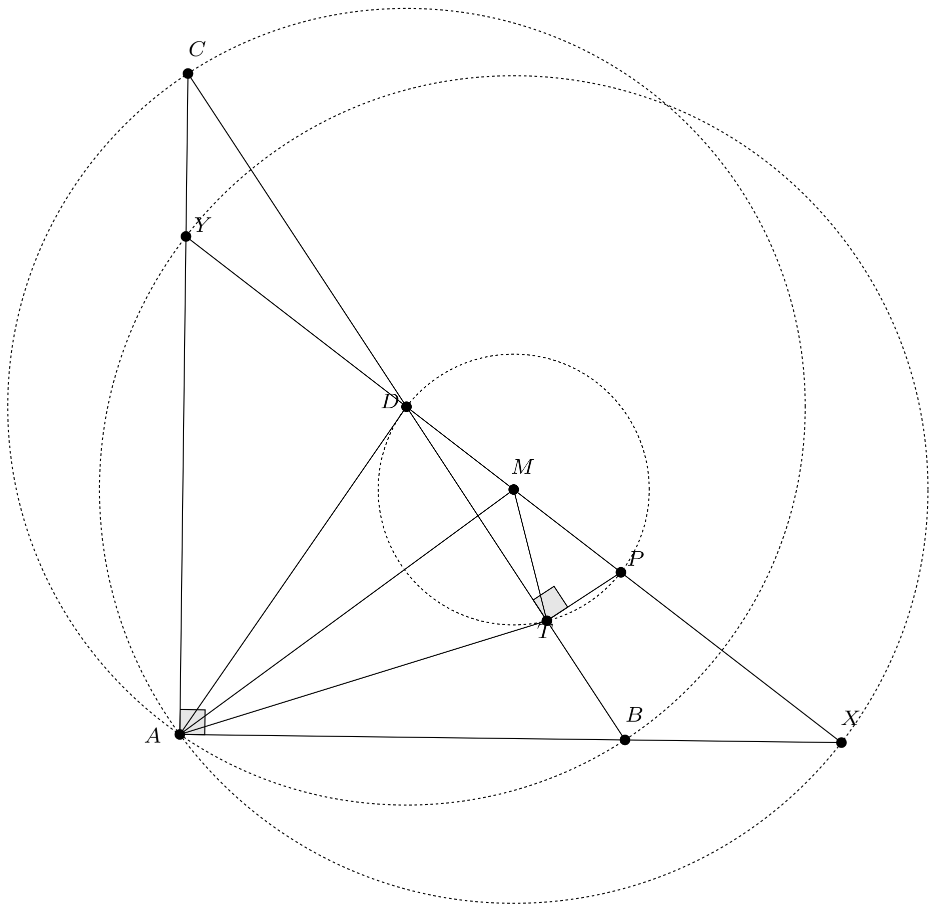

Figure 1: Right angles and congruent segments are an angle-chaser's dream come true!

Solution (Andrew Paul): Let $\angle DAM = \alpha$, $\angle MAT = \beta$, and $\angle TAB = \gamma$. We wish to show $\alpha = \beta$.

We have three prominent right triangles so naturally we draw their circumcircles. Recall that the hypotenuse of a right triangle is a diameter of its circumcircle, hence D is the center of (ABC). Now we have $AD = BD$, the radius of (ABC), so $\triangle ABD$ is isosceles. Thus we have:

$$\angle DAB = \angle ABD = \alpha + \beta + \gamma$$

From this it follows that $\angle ACB = 90^\circ - (\alpha + \beta + \gamma)$. We note that $AM = XM$, the radius of (AXY) so $\triangle AMY$ is isosceles with:

$$\angle MAY = 90^\circ - (\beta + \gamma) = \angle AYM$$

We also have $\angle DYC = 180^\circ - \angle AYM = 90^\circ + \beta + \gamma$. So:

$$\angle CDY = \angle MDT = 180^\circ - (\angle DYC + \angle ACB) = 180^\circ - [90^\circ + \beta + \gamma + 90^\circ - (\alpha + \beta + \gamma)] = \alpha$$

Now $\angle MDT = \angle DTM = \alpha$ because $\triangle MDT$ is isosceles with congruent sides that are radii of (MDT).

We've chased α from $\angle DAM$ to $\angle DTM$ which is enough to imply that $MDAT$ is cyclic. Hence $\angle MAT = \angle MDT$ and $\alpha = \beta$ as desired. $\square$

Problem (Canada 1991/3): Let P be a point inside circle ω. Consider the set of chords of ω that contain P. Prove that the locus formed by the midpoints of these chords is a circle.

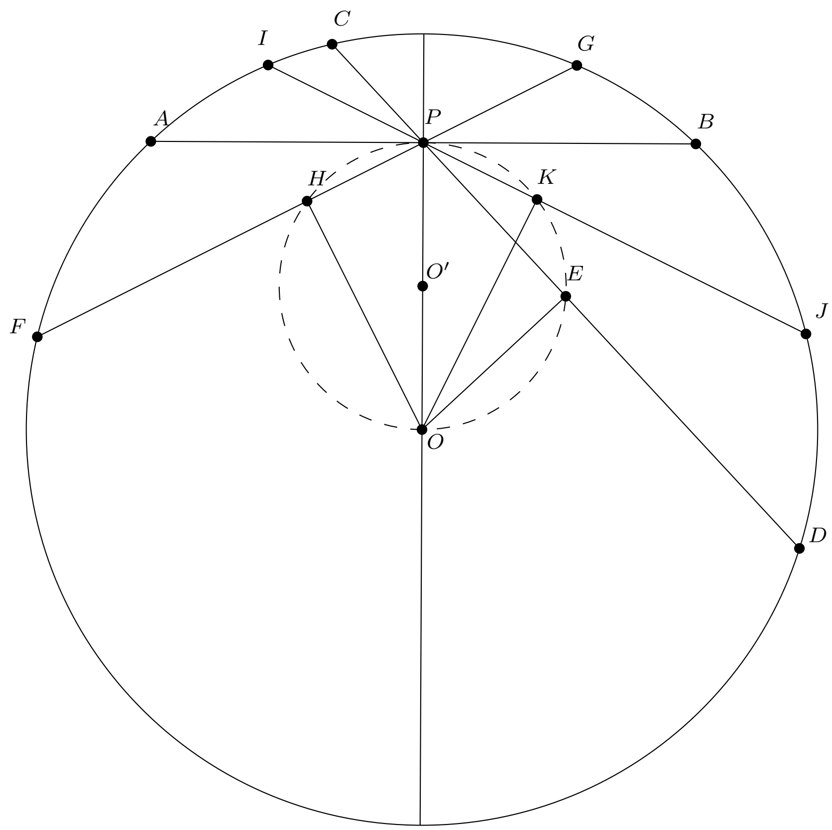

Figure 2: Here we let $X_1' \in \{G, J\}$ and $X_2' \in \{F, I\}$

Solution (Andrew Paul): We draw two special chords. Let $\overline{AB}$ be the chord containing P such that P is also the chord's midpoint. Now we draw the diameter of the circle that passes through P. The midpoint of the diameter is the center of ω which we will denote as O. From the conditions, it follows that O and P themselves must lie on the locus, which we will denote as ω'.

Draw a third chord, CD, and let its midpoint be E. By SSS, we find that $\triangle OEC \cong \triangle OED$ so $\angle OEC = \angle OED = \frac{180°}{2} = 90°$. It follows that if ω' is a circle, $\omega' = (PEO)$ and $\angle PEO = 90°$, so $\overline{OP}$ would be a diameter of ω'.

Suppose $X \in \omega' \backslash \{P, E, O\}$. For our result to be valid, P, E, O, and X must be concyclic. In other words, it suffices to show that $PEOX$ is cyclic. Let us define X_1' as the endpoint of the chord with midpoint X that is on the same side of the circle as E WRT the diameter through P, and X_2' as the other endpoint of the chord with midpoint X. We simply observe:

$$\triangle OXX_1' \cong \triangle OXX_2' \Rightarrow \angle OXX_1' = \angle OXX_2' = 90^\circ = \angle PEO$$

So $PEOX$ is cyclic as desired. $\square$