Problems on Magnetic Fields

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Problem 1: The current-carrying wire loop in the figure lies all in one plane and consists of a semicircle of radius 10.0 cm, a smaller semicircle with the same center, and two radial lengths. Thet smaller semicircle is rotated out of that plane by angle θ, until it is perpendicular to the plane. The graph of $B(\theta)$, the magnitude of the net magnetic field at the central point P shows that $B(90°) = 10.0\ \mu$T and $B(0) = 12.0\ \mu$T. What is the radius of the smaller semicircle?

Solution: Let us generalize and suppose that the larger semicircle has radius R and the smaller semicircle has radius r. Let the central point where we are measuring the magnetic field from be P. Let the magnitude of the net magnetic field when the small semicircle is deflected at an angle of θ be $B(\theta)$. Then, by the Biot-Savart law, the net magnetic field at P is given by

$$\begin{aligned} B(0) &= \frac{\mu_0}{4\pi}\left[\frac{1}{R^2}\int_0^{\pi R^2} i\ \mathrm{d}\ell + \frac{1}{r^2}\int_0^{\pi r^2} i\ \mathrm{d}\ell\right] \\ &= \frac{\mu_0 i}{4}\left(\frac{1}{R}+\frac{1}{r}\right). \end{aligned}$$

In this case, since the entirety of $\mathbf{B}$ exists in one component, there is no need to distinguish between $B(\theta)$ and $\mathbf{B}(\theta)$, the actual magnetic field at P for an angle of deflection θ. Let us now deflect the smaller semicircle to an arbitrary acute angle θ. Let L be the point on the smaller semicircle such that $\overline{PL}$ is perpendicular to the diameter of the larger semicircle.

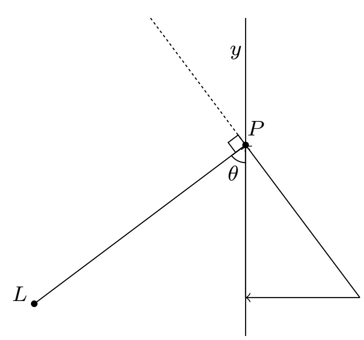

Figure 1: This diagram shows the portion of vector $\mathbf{B}(\theta)$ contributed by the smaller semicircle split into its y and z components

The part of $\mathbf{B}(\theta)$ contributed by the smaller semicircle, $\mathbf{B}_r(\theta)$ is thus:

$$\begin{aligned}\mathbf{B}_r(\theta) &= \left\langle 0, \frac{\mu_0}{4\pi r^2}\cos(90^\circ - \theta)\int_0^{\pi r^2} i \; \mathrm{d}\ell, \frac{\mu_0}{4\pi r^2}\sin(90^\circ - \theta)\int_0^{\pi r^2} i \; \mathrm{d}\ell \right\rangle \\ &= \frac{\mu_0 i}{4}\left\langle 0, \frac{1}{r}\sin\theta, \frac{1}{r}\cos\theta \right\rangle.\end{aligned}$$

The portion of $\mathbf{B}(\theta)$ contributed by the larger semicircle, $\mathbf{B}_R(\theta)$ does not change. It is,

$$\begin{aligned}\mathbf{B}_R(\theta) &= \left\langle 0, 0, \frac{\mu_0}{4\pi R^2}\int_0^{\pi R^2} i \; \mathrm{d}\ell \right\rangle \\ &= \frac{\mu_0 i}{4}\left\langle 0, 0, \frac{1}{R} \right\rangle,\end{aligned}$$

hence,

$$\begin{aligned}\mathbf{B}(\theta) &= \mathbf{B}_R(\theta) + \mathbf{B}_r(\theta) \\ &= \frac{\mu_0 i}{4}\left\langle 0, \frac{1}{r}\sin\theta, \frac{1}{R} + \frac{1}{r}\cos\theta \right\rangle.\end{aligned}$$

From this, we can compute $B(\theta)$:

$$\begin{aligned}B(\theta) &= \frac{\mu_0 i}{4}\sqrt{\frac{1}{r^2}\sin^2\theta + \left(\frac{1}{R} + \frac{1}{r}\cos\theta\right)^2} \\ &= \frac{\mu_0 i}{4}\sqrt{\frac{1}{R^2} + \frac{1}{r^2} + \frac{2}{Rr}\cos\theta}.\end{aligned}$$

This yields $B(90^\circ) = \frac{\mu_0 i}{4}\sqrt{\frac{1}{R^2} + \frac{1}{r^2}}$. We now have the system of equations

$$\begin{cases} B(0) = \frac{\mu_0 i}{4}\left(\frac{1}{R} + \frac{1}{r}\right) \\ B(90^\circ) = \frac{\mu_0 i}{4}\sqrt{\frac{1}{R^2} + \frac{1}{r^2}}. \end{cases}$$

Let $Q = \frac{B(90^\circ)}{B(0)}$, $K = R^{-1}$, and $\kappa = r^{-1}$. Dividing the two equations we obtain:

$$Q = \frac{\sqrt{K^2 + \kappa^2}}{K + \kappa},$$

which rearranges to

$$M(K, \kappa) = (Q^2 - 1)\kappa^2 + 2Q^2 K\kappa + (Q^2 - 1)K^2 = 0.$$

M can be viewed as quadratic in κ. The solutions, by the quadratic formula, are:

$$\kappa \in \left\{ \frac{-Q^2 \pm \sqrt{2Q^2 - 1}}{Q^2 - 1} K \right\}$$

There are two solutions here. How do we know which one is the correct solution and which is extraneous?

Observe that M is a multivariable polynomial that is symmetric in K and κ. That is, $M(K, \kappa) = M(\kappa, K)$. Hence the solution to K is identical in form to that of κ:

$$K \in \left\{ \frac{-Q^2 \pm \sqrt{2Q^2 - 1}}{Q^2 - 1} \kappa \right\}$$

Since we are required to have $K \neq \kappa$, we must choose distinct signs for $\pm$ in the solutions to K and κ (they cannot both be plus or both be minus as this would force $K = \kappa$, $K = \kappa = 0$, or $K = -\kappa$, all of which are invalid). Furthermore, we have the condition that $K < \kappa$, so we must choose the signs such that this inequality is satisfied.

Since choosing minus maximizes the magnitude of the numerator, enabling κ to be a larger multiple of K than K is of κ, we conclude that the only non-extraneous solution is:[1]

$$\kappa = -\frac{Q^2 + \sqrt{2Q^2 - 1}}{Q^2 - 1} K.$$

Plugging in the numbers, we find that $\boxed{r \approx 2.32 \text{ cm}}$.

Problem 2: Consider a square of side length a with a current i flowing through it. Find the net magnetic field $B(z)$, at a point that is z away from the square on the axis through its center and orthogonal to the plane of the square.

Solution: The Biot-Savart law states that:

$$\mathrm{d}\mathbf{B} = \frac{\mu_0}{4\pi} \cdot \frac{\mathbf{i} \times \mathrm{d}\hat{\mathbf{r}}}{r^2} \Rightarrow \mathrm{d}B = \frac{\mu_0 i}{4\pi} \cdot \frac{\sin\theta}{r^2}\, \mathrm{d}\ell,$$

so it suffices to compute the distance between the $(0, 0, z)$ and points on the square and the angle between the current vector and the radial vector.

Center the square at the origin. Drawing the midpoints of the square, we can partition it into eight congruent segments. By symmetry, the magnetic field at $(0, 0, z)$ is equal to eight times the magnetic field contributed by one of these segments. WLOG, let us consider the segment from $\left(\frac{a}{2}, 0, 0\right)$ to the vertex $\left(\frac{a}{2}, \frac{a}{2}, 0\right)$.

A point on this segment, $P(\ell)$, is of the form $\left(\frac{a}{2}, \ell, 0\right)$ for $0 \leq \ell \leq \frac{a}{2}$. The distance from $(0, 0, z)$

[1] Don't worry about the overall sign. Both solutions *must* be positive due to the physical ramifications of this problem. Though you could also prove this mathematically by noting $Q < 1$.

to $P(\ell)$ is given by:

$$r(z,\ell) = \sqrt{\frac{a^2}{4} + z^2 + \ell^2},$$

to find the angle, we compute the radial vector from $(0,0,z)$ to $P(\ell)$, which is $\mathbf{r} = \left[\frac{a}{2}, \ell, -z\right]$. The current vector is $\mathbf{i} = [0, i, 0]$. Then,

$$\begin{aligned}\cos\theta &= \frac{\mathbf{r}\cdot\mathbf{i}}{|\mathbf{r}||\mathbf{i}|}\\ &= \frac{\ell i}{ri}\\ &= \frac{\ell}{r}.\end{aligned}$$

Now,

$$\begin{aligned}\sin\theta &= \sqrt{1-\cos^2\theta}\\ &= \sqrt{1-\frac{\ell^2}{r^2}}\\ &= \frac{1}{r}\sqrt{\frac{a^2}{4}+\ell^2}.\end{aligned}$$

Multiplying the Biot-Savart law by eight, we obtain:

$$\begin{aligned}B(z) &= \frac{2\mu_0 i}{\pi}\sqrt{\frac{a^2}{4}+z^2}\int_0^{\frac{a}{2}} \frac{1}{r^3}\ \mathrm{d}\ell\\ &= \frac{2\mu_0 i}{\pi}\cdot\frac{1}{\frac{a^2}{4}+z^2}\int_0^{\frac{a}{2}}\left(1+\frac{\ell^2}{\frac{a^2}{4}+z^2}\right)^{-\frac{3}{2}}\ \mathrm{d}\ell.\end{aligned}$$

We make the trigonometric substitution $\ell = \tan\theta\sqrt{\frac{a^2}{4}+z^2}$. Then, $\mathrm{d}\ell = \sec^2\theta\sqrt{\frac{a^2}{4}+z^2}\ \mathrm{d}\theta$, and our integral becomes

$$B(z) = \frac{2\mu_0 i}{\pi}\cdot\frac{1}{\sqrt{\frac{a^2}{4}+z^2}}\int_0^{\theta_f}\cos\theta\ \mathrm{d}\theta,$$

where $\theta_f = \arctan\frac{a}{2\sqrt{\frac{a^2}{4}+z^2}}$.

The rest is just computation. This evaluates to

$$\boxed{B(z) = \frac{2\mu_0 i\sqrt{2}}{\pi}\cdot\frac{a}{\sqrt{(a^2+4z^2)(a^2+2z^2)}}}.$$

Problem 3: Consider a railgun.

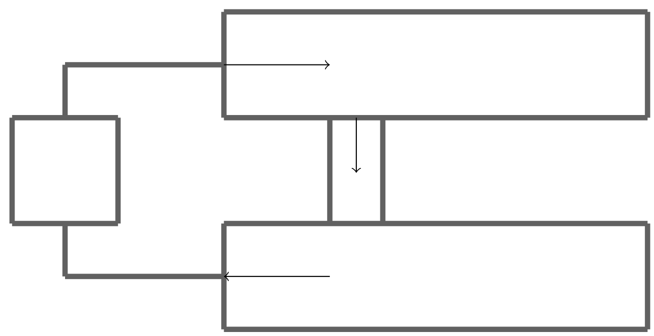

Figure 2: The square is a power source, the rectangles are cylindrical rails, and i is shown

The railgun consists of a power source attached to two cylindrical rails of radius R and a conducting projectile bridging the gap between the rails. Let the current i be directed as shown and let the gap spanned between the two rails be w. Find the force of the railgun on the projectile.

Solution: It is important to recognize that both rails comprise of a single system. That is current cannot flow through one if it does not flow through the other. This means that each rail does not contribute its own independent magnetic field. Rather, the field contributed by either one when isolated from the system is the same as the net field in the system.[2]

This means it suffices to find the field contributed by one of the rails. Due to symmetry, the tangential vectors and field vectors are parallel for circular Amperian loops, and so Ampere's law for the magnetic field due to the cylindrical rail with a current i through it at a distance $r > R$ reduces to

$$\oint \mathbf{B} \cdot \mathrm{d}\ell = \int_0^{2\pi r} B \ \mathrm{d}\ell = \mu_0 i \Rightarrow B = \frac{\mu_0 i}{2\pi r}.$$

In our case, $r = R + \ell$ where $0 \leq \ell \leq w$ is the distance down the projectile we are considering. We also have,

$$\mathrm{d}\mathbf{F} = \mathrm{d}\mathbf{i} \times \mathbf{B} \Rightarrow \mathrm{d}F = iB \sin\theta \ \mathrm{d}\ell.$$

Since the current through the projectile is orthogonal to the field, $\sin\theta = 1$ and $\mathrm{d}F = iB \ \mathrm{d}\ell$.

[2]This phenomenon is analogous to why the field between two charged conducting parallel plates is deduced by considering each to a be a sheet of charge instead of a conductor. The field of one charged plate polarizes the other so there is no field individually contributed by each plate, rather the net field is already a result of the two. *The charge on one of the plates is the result of the electric field, not the other way around.* See here: https://mathwithandy.weebly.com/blog/capacitance

Therefore,

$$\begin{aligned} F &= \int \mathrm{d}F \\ &= \frac{\mu_0 i^2}{2\pi} \int_0^w \frac{\mathrm{d}\ell}{R+\ell} \\ &= \frac{\mu_0 i^2}{2\pi} \left[\log{(w+R)} - \log R\right] \\ &= \boxed{\frac{\mu_0 i^2}{2\pi} \log \frac{w+R}{R}}. \end{aligned}$$

The right-hand rules imply that this force is directed to the right as expected.