Inertial Frames of Reference and Motion in Polar Coordinates

Andrew Paul

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We begin by defining an *inertial frame of reference*. This is simply a frame of reference from which the particle being observed possesses no acceleration. For example, for a particle in free fall, an inertial frame of reference would be that of the particle itself. Alternatively, one could also consider the frame of reference of someone falling alongside the particle. Or for that matter, one may consider the frame of reference of someone falling above or below the particle. As long as both the particle and the frame are in free fall, subject to the same acceleration, the acceleration between the frame and the particle itself is zero, hence that frame must be inertial.

This naturally leads us to define a *non-inertial frame of reference* as one that has an acceleration relative to an inertial frame. These frames are not to be confused with frames that are moving at constant velocities with respect to each other, thereby causing observed acceleration to be equivalent across both frames. If a frame happens to be moving at constant velocity with respect to an inertial frame, it too must also be an inertial frame.

While an observer in an inertial frame will measure an acceleration of zero, an observer in a non-inertial frame will *not*, and to this observer, the particle under observation is under the effects of what are referred to as *fictitious forces*. We will briefly derive one of these, known as the Coriolis force (which causes the Coriolis effect), before moving on to discuss other problems of motion in polar coordinates. These problems are pulled from exercises in *Introductory Classical Mechanics* by David Morin.

We begin by considering a vector from the origin to a point with cartesian coordinates (x, y). Such a vector, $\mathbf{r}$, would take the form:

$$\begin{aligned}\mathbf{r} &= x\hat{\mathbf{x}} + y\hat{\mathbf{y}} \\ &= r\cos\theta\hat{\mathbf{x}} + r\sin\theta\hat{\mathbf{y}}.\end{aligned}$$

To compute the force on a particle at the position vector $\mathbf{r}$, we must first find the acceleration vector. This is the second time derivative of $\mathbf{r}$. We take the first derivative, denoting derivatives with Newton's notation:

$$\dot{\mathbf{r}} = (\dot{r}\cos\theta - r\dot{\theta}\sin\theta)\hat{\mathbf{x}} + (\dot{r}\sin\theta + r\dot{\theta}\cos\theta)\hat{\mathbf{y}}.$$

Note that we use the fact that $\dot{\hat{\mathbf{x}}} = \dot{\hat{\mathbf{y}}} = 0$, because $\hat{\mathbf{x}}$ and $\hat{\mathbf{y}}$ are standard basis vectors, and thus constant. Taking another derivative:

$$\ddot{\mathbf{r}} = (\ddot{r}\cos\theta - 2\dot{r}\dot{\theta}\sin\theta - r\ddot{\theta}\sin\theta - r\dot{\theta}^2\cos\theta)\hat{\mathbf{x}} + (\ddot{r}\sin\theta + 2\dot{r}\dot{\theta}\cos\theta + r\ddot{\theta}\cos\theta - r\dot{\theta}^2\sin\theta)\hat{\mathbf{y}}.$$

We find that the acceleration vector takes on a much more natural form if we use polar standard basis vectors, rather than Cartesian. The construction is shown here:

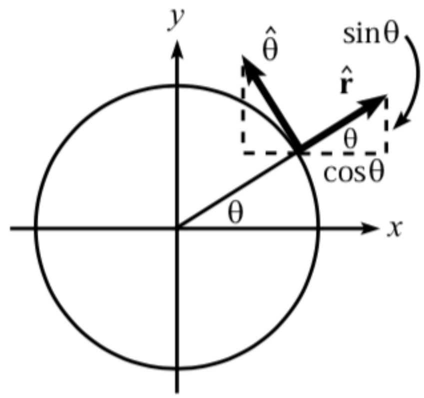

Figure 1: This diagram from *Introductory Classical Mechanics* shows the construction of the polar standard basis vectors.

This construction is quite intuitive. We want the vectors point in the directions in which a particle would move when subjected to a slight change in the parameter associated to them. By simple geometry, our construction implies:

$$\hat{\mathbf{r}} = \cos\theta\hat{\mathbf{x}} + \sin\theta\hat{\mathbf{y}},$$

$$\hat{\boldsymbol{\theta}} = -\sin\theta\hat{\mathbf{x}} + \cos\theta\hat{\mathbf{y}}.$$

Incidentally, observe that:

$$\frac{\partial\hat{\mathbf{r}}}{\partial\theta} = \hat{\boldsymbol{\theta}},$$

which makes sense since we wish for $\hat{\boldsymbol{\theta}}$ to be the direction in which the position vector will change when subjected to a slight change in angle.

Now we observe that we may rewrite $\ddot{\mathbf{r}}$ as:

$$\begin{aligned}
\ddot{\mathbf{r}} &= (\ddot{r}\cos\theta - 2\dot{r}\dot{\theta}\sin\theta - r\ddot{\theta}\sin\theta - r\dot{\theta}^2\cos\theta)\hat{\mathbf{x}} + (\ddot{r}\sin\theta + 2\dot{r}\dot{\theta}\cos\theta + r\ddot{\theta}\cos\theta - r\dot{\theta}^2\sin\theta)\hat{\mathbf{y}} \\
&= (\ddot{r}\cos\theta - r\dot{\theta}^2\cos\theta)\hat{\mathbf{x}} + (\ddot{r}\sin\theta - r\dot{\theta}^2\sin\theta)\hat{\mathbf{y}} - (2\dot{r}\dot{\theta}\sin\theta + r\ddot{\theta}\sin\theta)\hat{\mathbf{x}} + (2\dot{r}\dot{\theta}\cos\theta + r\ddot{\theta}\cos\theta)\hat{\mathbf{y}} \\
&= (\ddot{r} - r\dot{\theta}^2)[\cos\theta\hat{\mathbf{x}} + \sin\theta\hat{\mathbf{y}}] + (2\dot{r}\dot{\theta} + r\ddot{\theta})[-\sin\theta\hat{\mathbf{x}} + \cos\theta\hat{\mathbf{y}}] \\
&= (\ddot{r} - r\dot{\theta}^2)\hat{\mathbf{r}} + (2\dot{r}\dot{\theta} + r\ddot{\theta})\hat{\boldsymbol{\theta}}.
\end{aligned}$$

We define the total force to be the sum of forces along the standard basis vectors, $\mathbf{F} = F_r\hat{\mathbf{r}} + F_\theta\hat{\boldsymbol{\theta}}$.

Therefore:

$$F_r = m(\ddot{r} - r\dot{\theta}^2),$$

$$F_\theta = m(2\dot{r}\dot{\theta} + r\ddot{\theta}).$$

Nearly every term here is fairly intuitive. We can simply consider purely radial motion (in which case $\dot{\theta} = 0$) or purely circular motion (in which case $\dot{r} = 0$), and the equations above will yield well-known identities. For example, in circular motion, at least on the radial side of things, we obtain:

$$F_r = -mr\dot{\theta}^2 = -mr\omega^2 = -\frac{mv^2}{r},$$

which is simply centripetal force.

The outlier seems to be the $2m\dot{r}\dot{\theta}$ term. This term is the *Coriolis force.* Observe that it only exists if the particle is not moving purely circularly (in which case $\dot{r} = 0$) and is not moving purely radially (in which case $\dot{\theta} = 0$).

At this point, one may wonder *why* the Coriolis force appears in this derivation. Fictitious forces are only observed in non-inertial reference frames. Were we not working in an inertial reference frame?

The trick here is to define an inertial reference frame, S, from which we observe that the particle has no angular velocity or radial velocity. In this inertial frame, the particle is stationary. Our coordinate system (in which we did the above derivation), S', then possesses an angular velocity of $\dot{\theta}$ and a linear velocity of $\dot{r}$ with respect to the inertial frame S, excluding sign. From the vantage point of S', S *dances* in synchronization with the particle, rotating and translating as the particle does, such that the particle's coordinate in S never changes. So in fact, we were working in a non-inertial reference frame all along!

A more detailed discussion on the Coriolis force itself is outside the scope of this paper. For now, we continue with some problems.

Problem: Consider a particle that feels an angular force only, of the form $F_\theta = 2m\dot{r}\dot{\theta}$. Show that the trajectory takes the form of an exponential spiral, that is, $r = Ae^\theta$.

Solution: Since there is only angular force, $F_r = 0$, hence:

$$\ddot{r} = r\dot{\theta}^2.$$

The given condition on angular force provides:

$$\ddot{\theta} = 0 \Rightarrow \dot{\theta} = \int 0 \text{ d}t = C_1 \Rightarrow \theta = \int C_1 \text{ d}t = C_1 t + C_2.$$

Hence, our first equation becomes:

$$\ddot{r} = C_1^2 r.$$

The solution to this differential equation is simply $r = e^{C_1 t}$. Now, we can force some manipulations:

$$\begin{aligned} r &= e^{C_1 t} \\ &= e^{-C_2} \cdot e^{C_1 t + C_2} \\ &= Ae^{\theta}, \end{aligned}$$

for some constant A, as desired. $\square$

Problem: Consider a particle that feels an angular force only, of the form $F_\theta = 3m\dot{r}\dot{\theta}$. Show that $\dot{r} = \sqrt{Ar^4 + B}$. Also, show that the particle reaches $r = \infty$ in a finite time.

Solution: Since there is once again only angular force,

$$\ddot{r} = r\dot{\theta}^2.$$

However, this time our condition on angular force provides:

$$r\ddot{\theta} = \dot{r}\dot{\theta}.$$

For simplicity, let $\varphi = \dot{\theta}$. Then:

$$r\dot{\varphi} = \dot{r}\varphi.$$

We may separate variables and integrate:

$$\int \frac{1}{\varphi}\,\mathrm{d}\varphi = \int \frac{1}{r}\,\mathrm{d}r,$$

which yields $\varphi = Cr$. Substituting this in the first equation, we obtain:

$$\ddot{r} = Cr^3.$$

Integrating both sides with respect to r, the RHS becomes Cr^4. However, the LHS becomes:

$$\int \frac{\mathrm{d}^2 r}{\mathrm{d}t^2}\,\mathrm{d}r.$$

This becomes clearer if we let $u = \dot{r}$. Then, the integral is simply:

$$\int u\,\mathrm{d}u = \frac{1}{2}u^2 + K = Cr^4,$$

or simply:

$$\boxed{\dot{r} = \sqrt{Ar^4 + B}},$$

for constants A and B, as desired.

Showing that the particle reaches $r = \infty$ in a finite amount of time is considerably harder. I adapt a solution by Mathematics StackExchange user Sangchul Lee, used to show that the function satisfying $y' = 1 + y^4$ blows up in finite time.

Consider the ODE:

$$\dot{r} = \sqrt{Ar^4 + B} = \sqrt{B}\sqrt{\frac{A}{B}r^4 + 1}.$$

Euler's method shows that r must be concave up as there is a positive feedback loop-type effect. $\dot{r}$ is a strictly increasing function of r, so when r increases, $\dot{r}$ increases, causing r to increase further, ad infinitum. So, there must be t_c such that $\dot{r} \geq 1 \forall t \in \text{Dom } r \ni t \geq t_c$. Integrating this inequality, we obtain that for sufficiently large t:

$$r(t) \geq t.$$

Clearly, $\dot{r}$ is bounded by:

$$\dot{r} \geq r^2\sqrt{A} \Rightarrow \frac{\dot{r}}{r^2} \geq \sqrt{A}.$$

Suppose $t_c < t_0 < t_1$. We integrate both sides of the inequality above, with respect to t from t_0 to t_1:

$$\int_{t_0}^{t_1} \frac{\dot{r}}{r^2} \, \mathrm{d}t \geq \sqrt{A}(t_1 - t_0).$$

The LHS transforms via substitution:

$$\int_{t_0}^{t_1} \frac{\dot{r}}{r^2} \, \mathrm{d}t = \int_{r(t_0)}^{r(t_1)} \frac{1}{r^2} \, \mathrm{d}r = \frac{1}{r(t_0)} - \frac{1}{r(t_1)}.$$

Hence:

$$\frac{1}{r(t_0)} - \frac{1}{r(t_1)} \geq \sqrt{A}(t_1 - t_0).$$

But since $r(t) \geq t$, we must have $\frac{1}{t_0} \geq \frac{1}{r(t_0)} \geq \frac{1}{r(t_0)} - \frac{1}{r(t_1)}$, so:

$$\frac{1}{t_0} \geq \frac{1}{r(t_0)} - \frac{1}{r(t_1)} \geq \sqrt{A}(t_1 - t_0).$$

Now we simply observe that if we let $t_1 = t_0 + 1$ and let $t_0 \to \infty$, we must have $\frac{1}{t_0} \to 0$ but $\sqrt{A}(t_1 - t_0) \to \sqrt{A} > 0$, a contradiction.

Therefore, t_0 cannot be made arbitrarily large, so the domain of r must be restricted to some upper bound. The physical interpretation of r prohibits discontinuities, so a vertical asymptote must instead exist and so r blows up in finite time, as desired. $\square$

Reference:

Sangchul Lee's proof that $y' = 1 + y^4$ blows up in finite time:
https://math.stackexchange.com/a/710535/419177