Measures of a Hypersphere

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In this paper, we discuss the 4-dimensional hypersphere (referred to as a *glome* or a 3-sphere). We will derive an equation for the 4-dimensional hypervolume of a glome with radius r. Single-variable calculus will be used extensively in this discussion, in particular, integral calculus. It is expected that the reader can derive the volume of a sphere ($\frac{4\pi}{3}r^3$) easily using calculus.

Note that this is not by any means an original work of mine. This paper serves as a purely instructive resource compiled from numerous other sources, not as research or other original work of my own. The general layout of the proof is my own, though it is not unique, and much of the computation was verified by both *AoPS: Calculus* and *WolframAlpha*. The proof of the lemma that will be used is from *Wikipedia*. All links will be provided below.

First, *what is a glome*? It is the 4-dimensional analogue of the sphere and circle. Note that a circle has equation:

$$x^2 + y^2 = r^2$$

and a sphere has equation:

$$x^2 + y^2 + z^2 = r^2$$

Now letting a point in 4-dimensional Euclidean space (which we will abbreviate as 4-space) be represented as (w, x, y, z) such that the w-axis is the fourth axis that is orthogonal to the other three, it seems that a glome can be defined:

$$w^2 + x^2 + y^2 + z^2 = r^2$$

simply by *dimensional analogy* (an important heuristic technique that we will rely on to make an inductive hypothesis and other key observations later). To prove this with certainty, we simply consider 4-dimensional vector space ($\mathbb{R}^4$). A vector in this space is of the form:

$$\mathbf{v} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k} + a_4\mathbf{l}$$

where $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$, and $\mathbf{l}$ are the standard basis vectors in $\mathbb{R}^4$. The norm of $\mathbf{v}$ is:

$$||\mathbf{v}|| = \sqrt{a_1^2 + a_2^2 + a_3^2 + a_4^2}$$

This implies that the distance between (w, x, y, z) and the origin in 4-space is $\sqrt{w^2+x^2+y^2+z^2}$. By the definition of a glome, we set this equal to r:

$$\sqrt{w^2+x^2+y^2+z^2} = r \Rightarrow w^2+x^2+y^2+z^2 = r^2$$

as desired. This definition of an n-ball can easily be extended to any dimension n using a trivial induction.

Now that we know what a glome's equation looks like, we can think about how to find its 4-volume. Consider a sphere passing through a plane. On the plane, cross-sections are taken. First, when the plane is tangent to the sphere, there is a single point where both coincide. Then on the plane, a circular cross-section appears, grows to the great circle of the sphere, then decreases back to a point. By dimensional analogy, if a glome passes through 3-space, then we should be able to see a point, followed by a sphere expanding from this point until it reaches a certain maximum, and then decrease again until it converges to and vanishes from a point. That is, the cross-sections of a glome made by 3-space must be spheres. We can see this if we subtract a squared term from our glome equation:

$$w^2+y^2+z^2 = r^2 - x^2$$

So as x goes from $-r$ to r, we see that we have a sphere of radius $\sqrt{r^2-x^2}$ in the wyz-space. Hence our suspicion was correct. Then, letting $V_n(r)$ be the n-volume of an n-ball with radius r, it follows that our 4-volume must be:

$$V_4(r) = \frac{4\pi}{3}\int_{-r}^{r}\left(\sqrt{r^2-x^2}\right)^3 dx$$

Exploiting the the symmetry of the glome, we can simplify the limits of integration and compensate by multiplying by 2:

$$V_4(r) = \frac{8\pi}{3}\int_{0}^{r}\left(\sqrt{r^2-x^2}\right)^3 dx$$

But now what? It is not at all clear how to evaluate the integral. In fact, that integral is extremely difficult to compute.

Instead of trying to attack the integral, let's take a step back and think. A little bit of thinking brings us back to our trusty old friend: dimensional analogy.

Notice that the area of a circle is πr^2 and the volume of a sphere is $\frac{4\pi}{3}r^3$. In general, it seems that:

$$V_n(r) = \delta_n r^n$$

Where $\delta_n = V_n(1)$.

Lemma:

$$V_n(r) \propto r^n$$

Proof: We prove this using induction and set the above proportion as our inductive hypothesis.

The hypothesis clearly holds for the base case $n = 0$. Now suppose that the hypothesis is true for dimension $n = k$. Then we must have:

$$V_{k+1}(r) = \int_{-r}^{r} V_k\left(\sqrt{r^2 - x^2}\right) dx = \int_{-r}^{r} V_k\left(r\sqrt{1 - \left(\frac{x}{r}\right)^2}\right) dx$$

Recall that we are assuming $V_k(r) = \delta_k r^k$ so $V_k(r_1 r_2) = \delta_k r_1^k r_2^k$. This means that we can eliminate the extra factor of r in the argument of the function V_k by multiplying it by r^k. Hence, we have:

$$V_{k+1}(r) = r^k \int_{-r}^{r} V_k\left(\sqrt{1 - \left(\frac{x}{r}\right)^2}\right) dx$$

Now suppose we let $u = \frac{x}{r}$. Then we have $du = \frac{1}{r}\,dx$ so our integral becomes:

$$V_{k+1}(r) = r^{k+1} \int_{-1}^{1} V_k\left(\sqrt{1 - u^2}\right) du$$

But we notice that $V_{k+1}(1) = \int_{-1}^{1} V_k\left(\sqrt{1 - u^2}\right) du$. Therefore:

$$V_{k+1}(r) = r^{k+1} V_{k+1}(1)$$

Which completes our induction.

$\square$

The implications of our lemma are clear. Instead of computing $V_4(r)$, we can compute the equivalent expression $V_4(1)r^4$. We have:

$$V_4(1) = \frac{8\pi}{3} \int_0^1 \left(\sqrt{1 - x^2}\right)^3 dx$$

Aha! This integral looks much more friendly! The radical and its radicand screams at us to perform a trigonometric substitution. Who are we to hold back? Letting $x = \sin\theta$ gives $dx = \cos\theta\,d\theta$ and:

$$V_4(1) = \frac{8\pi}{3} \int_0^{\frac{\pi}{2}} \cos\theta \left(\sqrt{1 - \sin^2\theta}\right)^3 d\theta = \frac{8\pi}{3} \int_0^{\frac{\pi}{2}} \cos^4\theta\,d\theta$$

Now we use the double-angle identity $\cos^2\theta = \frac{\cos 2\theta + 1}{2}$ to obtain:

$$V_4(1) = \frac{2\pi}{3} \int_0^{\frac{\pi}{2}} (\cos 2\theta + 1)^2\,d\theta = \frac{2\pi}{3} \int_0^{\frac{\pi}{2}} \cos^2 2\theta + 2\cos 2\theta + 1\,d\theta$$

We apply it another time on the term $\cos^2 2\theta$ and also knock out the last two terms:

$$V_4(1) = \frac{2\pi}{3}\left((\sin 2\theta + \theta)\Big|_0^{\frac{\pi}{2}} + \int_0^{\frac{\pi}{2}} \cos^2 2\theta\,d\theta \right)$$

We apply double-angle again to get:

$$V_4(1) = \frac{2\pi}{3}\left((\sin 2\theta + \theta)\Big|_0^{\frac{\pi}{2}} + \frac{1}{2}\int_0^{\frac{\pi}{2}} \cos 2\theta + 1\, d\theta\right)$$

Finally! Everything works out nicely:

$$V_4(1) = \frac{2\pi}{3}\left(\frac{5}{4}\sin 2\theta + \frac{3}{2}\theta\right)\Big|_0^{\frac{\pi}{2}} = \frac{2\pi}{3}\left(\frac{3\pi}{4}\right) = \frac{\pi^2}{2}$$

This means that the 4-volume of a unit glome is $\frac{\pi^2}{2}$. Combining this with our lemma, we can conclude that the 4-volume of a glome with radius r is given by:

$$\boxed{V_4(r) = \frac{\pi^2}{2}r^4}$$

Q.E.D.

Reflections and Retrospect:

In fact, the method used in this discussion can be easily used to find the n-volume of *any* n-ball (think about it; the integral always becomes susceptible to trigonometric subtitution regardless of the dimension we are working in as long as we let $r = 1$). Our lemma is the key that connects the unit n-ball's n-volume to the general n-volume of the n-ball.

In case you're interested, here are the first few values of the δ coefficients in the n-volume formulae:

$$\delta = \left\{1, 2, \pi, \frac{4\pi}{3}, \frac{\pi^2}{2}, \frac{8\pi^2}{15}, \frac{\pi^3}{6}, \frac{16\pi^3}{105}, \frac{\pi^4}{24}, \frac{32\pi^4}{945}, \frac{\pi^5}{120}, \ldots\right\}$$

Note that we start at dimension $n = 0$.

Also note that differentiating $V_n(r)$ gives the $(n-1)$-volume enclosed by the n-ball. In our case, the 3-volume enclosed by a glome with radius r is $\frac{d}{dr}\left(\frac{\pi^2}{2}r^4\right) = 2\pi^2 r^3$.

By the way, that scary looking definite integral that we avoided computing can now be seen to be $\frac{3\pi}{16}r^4$. While this by no means appears tame, it doesn't look too wild either. How about the indefinite integral?

Well as it turns out, that's an entirely different story:

$$\int \left(\sqrt{r^2 - x^2}\right)^3 dx = \frac{1}{8}\left(3r^4 \arctan\left(\frac{x}{\sqrt{r^2 - x^2}}\right) + x(5r^2 - 2x^2)\sqrt{r^2 - x^2}\right) + C$$

Yeah good thing we didn't bother with that. And as always, thank you *WolframAlpha* for quenching my thirst but sparing the pain.

Sources:

https://en.wikipedia.org/wiki/Volume_of_an_n-ball#The_volume_is_proportional_to_the_nth_power_of_the_radius https://en.wikipedia.org/wiki/Volume_of_an_n-ball#Low_dimensions

https://www.wolframalpha.com/input/?i=integrate+(r%5E2-x%5E2)%5E(3%2F2)dx

And of course, much thanks to *AoPS* for teaching me calculus.