Gravitational Field of a Thin Wire on the z-axis

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Consider an infinitely long wire along the z-axis with a constant linear mass density λ. Let us analyze force of gravity on a particle with mass m on the xy-plane at $(x, y, 0)$. I will keep my exposition segregated with distinct references to physical reasoning with the intention of showing the interplay between the math and the powerful tools that descend from it in physics.

First, we calculate the force on the particle. By Newton's law of universal gravitation

$$\mathrm{d}F = \frac{\mu}{r^2 + z^2}\,\mathrm{d}M,$$

where $\mu = Gm$ is the standard gravitational parameter of the particle and $r^2 + z^2$ is the square of the distance from the particle to an infinitesimal section of the wire at coordinate $(0, 0, z)$. Since $\lambda = \frac{\mathrm{d}M}{\mathrm{d}z}$ is a constant, this is simply

$$\mathrm{d}F = \frac{\mu\lambda}{r^2 + z^2}\,\mathrm{d}z.$$

Hence, the net force on the particle from the entire wire is

$$\begin{aligned} F &= \int_{-\infty}^{\infty} \frac{\mu\lambda}{r^2 + z^2}\,\mathrm{d}z \\ &= \frac{\mu\lambda}{r}\left[\lim_{z\to\infty}\left(\arctan\frac{z}{r}\right) - \lim_{z\to-\infty}\left(\arctan\frac{z}{r}\right)\right] \\ &= \frac{\mu\lambda\pi}{r}. \end{aligned}$$

Now this is an interesting result. This means a two-dimensional being whose universe is the xy-plane will not be able to see the full extent of the wire over the z-axis apart from the cross-section in her universe, yet will experience a force of gravity $\propto \frac{1}{r}$ toward that body!

Due to symmetry, the net force points radially inwards to the origin. Therefore, the vertical component of the force will be $-\sin\arctan\frac{y}{x} = -\frac{y}{r}$ and likewise the horizontal component of the force will be $-\frac{x}{r}$. So the gravitational field is

$$\boxed{\mathbf{F} = -\mu\lambda\pi\left\langle \frac{x}{r^2}, \frac{y}{r^2}\right\rangle}.$$

Now we compute the work done by field on the particle over various paths C in the xy-plane. The following problems are taken from an MIT pset.

Consider the path where C is the half-line $y = 1$, $x \geq 0$. We then have

$$\begin{aligned}
\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r} &= -\mu\lambda\pi \int_C \frac{x}{x^2+y^2}\ \mathrm{d}x + \frac{y}{x^2+y^2}\ \mathrm{d}y \\
&= -\mu\lambda\pi \int_0^\infty \frac{x}{1+x^2}\ \mathrm{d}x \\
&= -\frac{\mu\lambda\pi}{2} \lim_{x\to\infty} \log\left(1+x^2\right) \\
&= \boxed{-\infty}.
\end{aligned}$$

Hence it is impossible to go arbitrarily far away from the origin on the half-line $y = 1$, $x \geq 0$ with a finite amount of energy.

Next we consider the path C which is a circle of radius a centered about the origin, traversed counterclockwise. We can compute this immediately by noting that this scenario is not much different from a normal circular orbit. Since gravity is always radially directed and orthogonal to the tangential $\mathbf{r}$ vector, every dot product will be zero leading to a total of zero work done. We confirm this with the line integral, using the natural parameterization $x = a\cos\theta$, $y = a\sin\theta$.

$$\begin{aligned}
\oint_C \mathbf{F} \cdot \mathrm{d}\mathbf{r} &= -\mu\lambda\pi \oint_C \frac{x}{x^2+y^2}\ \mathrm{d}x + \frac{y}{x^2+y^2}\ \mathrm{d}y \\
&= -\frac{\mu\lambda\pi}{a} \int_0^{2\pi} 0\ \mathrm{d}\theta \\
&= \boxed{0},
\end{aligned}$$

as expected.

The last path we consider is the segment from $(0, 1)$ to $(1, 0)$. This is a subset of the line $y = 1 - x$. Similar to the first curve, we simply let x be our free parameter (essentially we choose the parameterization $x = t$). From this we obtain

$$\begin{aligned}
\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r} &= \mu\lambda\pi \int_C \frac{x}{x^2+y^2}\ \mathrm{d}x + \frac{y}{x^2+y^2}\ \mathrm{d}y \\
&= -\mu\lambda\pi \int_0^1 \frac{2x-1}{2x^2-2x+1}\ \mathrm{d}x \\
&= -\frac{\mu\lambda\pi}{2} \log\left(2x^2-2x+1\right)\Big|_0^1 \\
&= \boxed{0}.
\end{aligned}$$

Now we shall go about showing that the the field $\mathbf{F}$ is conservative. We know from physics that gravitational fields are conservative. Let us prove that the one we have is also conservative.

A field is conservative if and only if its two-dimensional curl is zero. We compute this for $\mathbf{F}$:

$$\begin{aligned}\nabla \times \mathbf{F} &= \frac{\partial}{\partial x}\left(\frac{y}{x^2+y^2}\right) - \frac{\partial}{\partial y}\left(\frac{x}{x^2+y^2}\right) \\ &= -\frac{2xy}{(x^2+y^2)^2} + \frac{2xy}{(x^2+y^2)^2} \\ &= 0,\end{aligned}$$

so $\mathbf{F}$ is conservative as expected.

Recall that every sufficiently well-behaved function $f : \mathbb{R}^2 \to \mathbb{R}$ has symmetry of its second derivatives, which means that the curl of its gradient field is zero. By Green's theorem, the field is also then conservative. As it turns out, the converse is also true. Every two dimensional conservative field is the gradient field of some potential function that has equivalent second derivatives. Not only is this condition necessary, but it is also sufficient. That is, conservative fields *are* gradient fields.

The first method employs the fundamental theorem for line integrals. Since $\mathbf{F}$ is conservative,

$$\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \int_C \nabla f \cdot \mathrm{d}\mathbf{r} = f(B) - f(A),$$

where A, B are the endpoints of C, in the order of traversal. We consider arbitrary endpoints, fixing one and varying the other. Suppose we fix A at (x_0, y_0) and let $B = (x, y)$. Due to the path-independence of line integrals in a conservative field, we can choose the path $C : A \to (x, y_0) \to B$, with each leg accomplished in line segments. Let $C_1 : A \to (x, y_0)$ and $C_2 : (x, y_0) \to B$. The line integral over C_1 is then[1]

$$\begin{aligned}\int_{C_1} \mathbf{F} \cdot \mathrm{d}\mathbf{r} &= -\mu\lambda\pi \int_{C_1} \frac{x}{x^2+y^2}\ \mathrm{d}x + \frac{y}{x^2+y^2}\ \mathrm{d}y \\ &= -\mu\lambda\pi \int_{x_0}^{x} \frac{x}{x^2+y_0^2}\ \mathrm{d}x \\ &= -\frac{\mu\lambda\pi}{2}\left[\log\left(x^2+y_0^2\right) - \log\left(x_0^2+y_0^2\right)\right].\end{aligned}$$

The line integral over C_2 is[1]

$$\begin{aligned}\int_{C_2} \mathbf{F} \cdot \mathrm{d}\mathbf{r} &= -\mu\lambda\pi \int_{C_2} \frac{x}{x^2+y^2}\ \mathrm{d}x + \frac{y}{x^2+y^2}\ \mathrm{d}y \\ &= -\mu\lambda\pi \int_{y_0}^{y} \frac{y}{x^2+y^2}\ \mathrm{d}y \\ &= -\frac{\mu\lambda\pi}{2}\left[\log\left(x^2+y^2\right) - \log\left(x^2+y_0^2\right)\right].\end{aligned}$$

[1]Please excuse the abuse of the notation $\int_c^x f\ \mathrm{d}x$.

Equating the sum of these two results to $f(B) - f(A)$, we obtain

$$\boxed{f(x,y) = -\mu\lambda\pi \log r + K}$$

for some constant K.

Alternatively, let $\mathbf{F} = P\mathbf{i} + Q\mathbf{j}$. Then $P = \frac{\partial f}{\partial x}$ and $Q = \frac{\partial f}{\partial y}$. Therefore,

$$\begin{aligned} f(x,y) &= -\mu\lambda\pi \int P \, \mathrm{d}x \\ &= -\mu\lambda\pi \int \frac{x}{x^2+y^2} \, \mathrm{d}x \\ &= -\frac{\mu\lambda\pi}{2} \log\left(x^2+y^2\right) + K_1(y) \\ &= -\mu\lambda\pi \log r + K_1(y). \end{aligned}$$

Similarly, using Q, we obtain $f(x,y) = -\mu\lambda\pi \log r + K_2(x)$. This implies that $K_1(y) = K_2(x)$. But the only way that two functions of separate independent variables can be equal is if those functions are constant. Hence,

$$\boxed{f(x,y) = -\mu\lambda\pi \log r + K}$$

for some constant K.

Can we do better? Can we determine the constant K? Let us reason physically.

The function f is the potential function, whose gradient field is the force field $\mathbf{F}$. That is, $\mathbf{F}$ is always orthogonal to the contours of f. This means that the potential function f *is* the gravitational potential energy function. We prove that the force field must be orthogonal to the potential function everywhere by contradiction.

Suppose to the contrary, $\mathbf{F}$ was not orthogonal to the contours of the potential function (equipotential surfaces). Then, there exists components of $\mathbf{F}$ along the surface. Therefore, it requires work to move a particle from A to B when both those points are on the equipotential surface. But then the potential at A cannot be equal to potential at B, contradiction. ■

But notice that this does not identify a unique potential function. There is an infinitely large family of functions, all off by a constant, that can be the potential function of the force field. This is because all that matters is the *shape* of the contours, not the potential energy value allocated to them. For instance, it makes no difference if I assign a value of 5 or 18 to a particular contour. As long as it has a particular shape, it will be orthogonal to the force field and could be defined as a potential function. You can visualize this by graphing $U = f(x,y)$ in the xyU-space. It does not matter if we translate the graph along the U-axis. The cross-sectional projections onto the xy-plane are invariant under such translations since they are isometric and have no x or y components.

In the standard scenario where the force of gravity is $\propto \frac{1}{r^2}$, gravitational potential energy, U, is often defined as the amount of work done by gravity on an object that moves from a position R

to ∞. That is,

$$U = -\int_R^\infty \mathbf{F} \cdot \mathrm{d}\mathbf{r} \propto \frac{1}{R}.$$

This works out well because $\lim_{r\to\infty} \int F \ \mathrm{d}r = 0$, when the constant of integration is taken to be 0. In defining U this way, observe that we are also simply setting it equivalent to the indefinite integral of F with the constant of integration taken to be 0. This is fine because the constant of integration vanishes anyway in the evaluation of a definite integral which shows the true energy difference between two different equipotential surfaces.

But in our case, we have a problem. Since we have $F \propto \frac{1}{r}$, the integral $-\int_R^\infty \mathbf{F} \cdot \mathrm{d}\mathbf{r}$ diverges. So instead, we take the second approach of definition that we discussed above. We simply define the gravitational potential energy to be $\int F \ \mathrm{d}r$ with the constant of integration taken to be zero. That is, we could really choose K to be *any* constant because gravitational potential energy is meaningless in our case unless a difference between two energies is taken in the evaluation of a definite integral.

In general, a force field $\mathbf{F}$ need not be symmetric about the origin. In such cases, integration with respect to r is useless. The potential energy functions then could be determined with the two methods we used above, namely with the fundamental theorem of line integrals and the fact that $\nabla \times \mathbf{F} = 0$. It just so happened that in our case, the potential function only depended on the quantity $\sqrt{x^2 + y^2} = r$, implying that our field $\mathbf{F}$ is symmetric about the origin.

We finish by using the fact that $f(x, y) = -\mu\lambda\pi \log r + K$ to show that $\int_C \mathbf{F} \cdot \mathrm{d}r$ for any path C joining P_1 and P_2 is dependent on the ratio $\frac{r_2}{r_1}$, where r_i is the distance of P_i from the origin. This follows immediately from the fundamental theorem of line integrals, which shows that the integral evaluates as

$$-\mu\lambda\pi \left(\log r_2 - \log r_1\right) = -\mu\lambda\pi \log \frac{r_2}{r_1}.$$