Total Time for Gravitational Collision With One Object Held Stationary

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Consider a planet A with mass M and radius R, and a second (pointlike) object B with mass $m \ll M$ released from some distance D away from the surface of A in space. How long does it take for the two objects to collide?

Note: In this problem, we will neglect the acceleration of A towards B, since $M \gg m$.

First, we define a coordinate system. Let A be centered at the origin, and let B start at the coordinate $(R + D, 0)$. We wish to find how long it takes for B to get from $(R + D, 0)$ to $(R, 0)$.

By the conservation of energy, the sum of the kinetic energy and the gravitational potential energy (GPE) is a constant. This constant is equal to the GPE of B right before it is released:

$$K + U = U_0$$

We have $K = \frac{1}{2}mv^2$, hence:

$$\frac{1}{2}mv^2 + U = U_0$$

To define the GPE, we must choose a reference point where the GPE is equal to 0. A natural candidate for such a reference point is the center of A, the origin. Then, we can define the GPE as the negative of the work done by gravity on an object in free fall from some coordinate $(x, 0)$ to the origin (or equivalently, the work done by some force to lift an object from the origin to $(x, 0)$):

$$U = \int_x^0 \frac{GMm}{x^2}\,\mathrm{d}x = -\int_0^x \frac{GMm}{x^2}\,\mathrm{d}x$$

Where the integrand follows from Newton's Law of Universal Gravitation. However, this choice of reference point presents a problem. The improper integral above diverges. This implies that the closer an object is to the center of gravity of a much more massive object, the harder it is to lift that object, until the object and center of gravity coincide with each other, in which case the task becomes impossible!

Instead, we choose a different reference point. In fact, we can choose *any* other reference point.

Conservation of energy will still yield the same result. Intuitively, this is because the initial GPE, U_0, will be calculated with respect to the same reference point as the instantaneous GPE, U. This is analogous to being able to let the height in the expression for GPE in a uniform gravitational field (mgh) be taken with respect to any vertical coordinate.

We can demonstrate this more rigorously. Suppose we choose the point on the surface of A that is the closest to B, namely $(R, 0)$, as our reference point. Then:

$$\frac{1}{2}mv^2 - \int_R^x \frac{GMm}{x^2}\ \mathrm{d}x = -\int_R^{R+D} \frac{GMm}{x^2}\ \mathrm{d}x$$

Performing the integration:

$$\frac{1}{2}mv^2 - \frac{GMm}{x} + \frac{GMm}{R} = \frac{GMm}{R} - \frac{GMm}{R+D}$$

The term dependent upon our reference point cancels! This proves that the resulting relationship between velocity and position is independent of our reference point. Observe that the relationship is not dependent on the mass of B either.

$$v = -\sqrt{2GM\left(\frac{1}{x} - \frac{1}{R+D}\right)}$$

We take the negative sign to account for the fact that we defined left to be negative. Note that this is a first-order nonlinear differential equation. The equation is in fact separable. We let $k = \sqrt{2GM}$ and $\alpha = (R+D)^{-1}$. Then we obtain:

$$-\int \frac{1}{\sqrt{\frac{1}{x} - \alpha}}\ \mathrm{d}x = \int k\ \mathrm{d}t$$

The RHS is trivially kt. The rest of the problem boils down to integrating the LHS:

$$I = -\int \frac{1}{\sqrt{\frac{1}{x} - \alpha}}\ \mathrm{d}x = kt$$

We tackle this by first substituting $u = \sqrt{\frac{1}{x} - \alpha}$. Then, $\mathrm{d}u = -\frac{1}{2x^2u}\ \mathrm{d}x$. Now our integral becomes:

$$I = 2\int x^2\ \mathrm{d}u$$

Rearranging our definition of u, we find $x = \frac{1}{\alpha+u^2}$, hence our integral becomes:

$$I = 2\int \frac{1}{(\alpha+u^2)^2}\ \mathrm{d}u = \frac{2}{\alpha^2}\int \frac{1}{\left(1+\frac{u^2}{\alpha}\right)^2}\ \mathrm{d}u$$

Next, we substitute $u = \sqrt{\alpha}\tan\theta$. Note that $\mathrm{d}u = \sqrt{\alpha}\sec^2\theta\ \mathrm{d}\theta$, and our integral becomes:

$$\begin{aligned} I &= 2\alpha^{-3/2}\int \frac{\sec^2\theta}{(1+\tan^2\theta)^2}\ \mathrm{d}\theta \\ &= 2\alpha^{-3/2}\int \cos^2\theta\ \mathrm{d}\theta \end{aligned}$$

Which is finally in familiar territory! The cosine double angle formula yields $\cos 2\theta = 2\cos^2\theta - 1$. Rearranging this to solve for the square of cosine and substituting this into our original integral leads to:

$$I = \alpha^{-3/2}\int 1 + \cos 2\theta\ \mathrm{d}\theta = \alpha^{-3/2}\left(\theta + \frac{1}{2}\sin 2\theta\right) + C$$

Finally, we reverse our substitutions. We have $\theta = \arctan\frac{u}{\sqrt{\alpha}}$. By the double angle formula for sine, we have $\frac{1}{2}\sin 2\theta = \sin\theta\cos\theta$. Since $\tan^2\theta + 1 = \sec^2\theta$, we have $\cos\theta = \frac{1}{\sqrt{1+\tan^2\theta}}$. Hence:

$$\cos\arctan\frac{u}{\sqrt{\alpha}} = \frac{1}{\sqrt{1+\frac{u^2}{\alpha}}} = \frac{\alpha}{\sqrt{\alpha^2+\alpha u^2}}$$

And:

$$\sin\arctan\frac{u}{\sqrt{\alpha}}\sqrt{1-\cos^2\arctan\frac{u}{\sqrt{\alpha}}} = \sqrt{\frac{\alpha^2+\alpha u^2-\alpha^2}{\alpha u^2+\alpha^2}} = \frac{u\sqrt{\alpha}}{\sqrt{\alpha^2+\alpha u^2}}$$

Hence:

$$\frac{1}{2}\sin 2\theta = \frac{u\alpha^{3/2}}{\alpha^2+\alpha u^2}$$

This yields:

$$I = \frac{u}{\alpha^2+\alpha u^2} + \alpha^{-3/2}\arctan\frac{u}{\sqrt{\alpha}} + C$$

Reversing our $u - \textit{substitution}$:

$$I = \alpha^{-3/2}\left(\sqrt{\alpha x - \alpha^2 x^2} + \arctan\sqrt{\frac{1}{\alpha x} - 1}\right) + C$$

And now we undo our $\alpha = (R+D)^{-1}$ and our $k = \sqrt{2GM}$ substitutions to obtain:

$$t = \sqrt{\frac{(R+D)^3}{2GM}}\left(\sqrt{\frac{x}{R+D} - \frac{x^2}{(R+D)^2}} + \arctan\sqrt{\frac{R+D}{x} - 1}\right) + C$$

To find the constant of integration, C, we note the initial condition of $x = R+D$ at $t = 0$. This yields $C = 0$. We have thus derived our main result:

$$\boxed{t = \sqrt{\frac{(R+D)^3}{2GM}}\left(\sqrt{\frac{x}{R+D} - \frac{x^2}{(R+D)^2}} + \arctan\sqrt{\frac{R+D}{x} - 1}\right)}$$

We have solved the problem, but we have not yet answered the question. When does the collision occur? It occurs when $x = R$. This gives us:

$$\boxed{t_f = \frac{1}{\sqrt{2GM}}\left((R+D)\sqrt{\frac{RD}{R+D}} + (R+D)^{3/2}\arctan\sqrt{\frac{D}{R}}\right)}$$

Amazing! Observe that $\lim_{R\to 0} t = \frac{1}{\sqrt{2GM}}\left(D\sqrt{x - \frac{x^2}{D}} + D^{3/2}\arctan\sqrt{\frac{D}{x} - 1}\right)$ describes the case where A is a point mass.

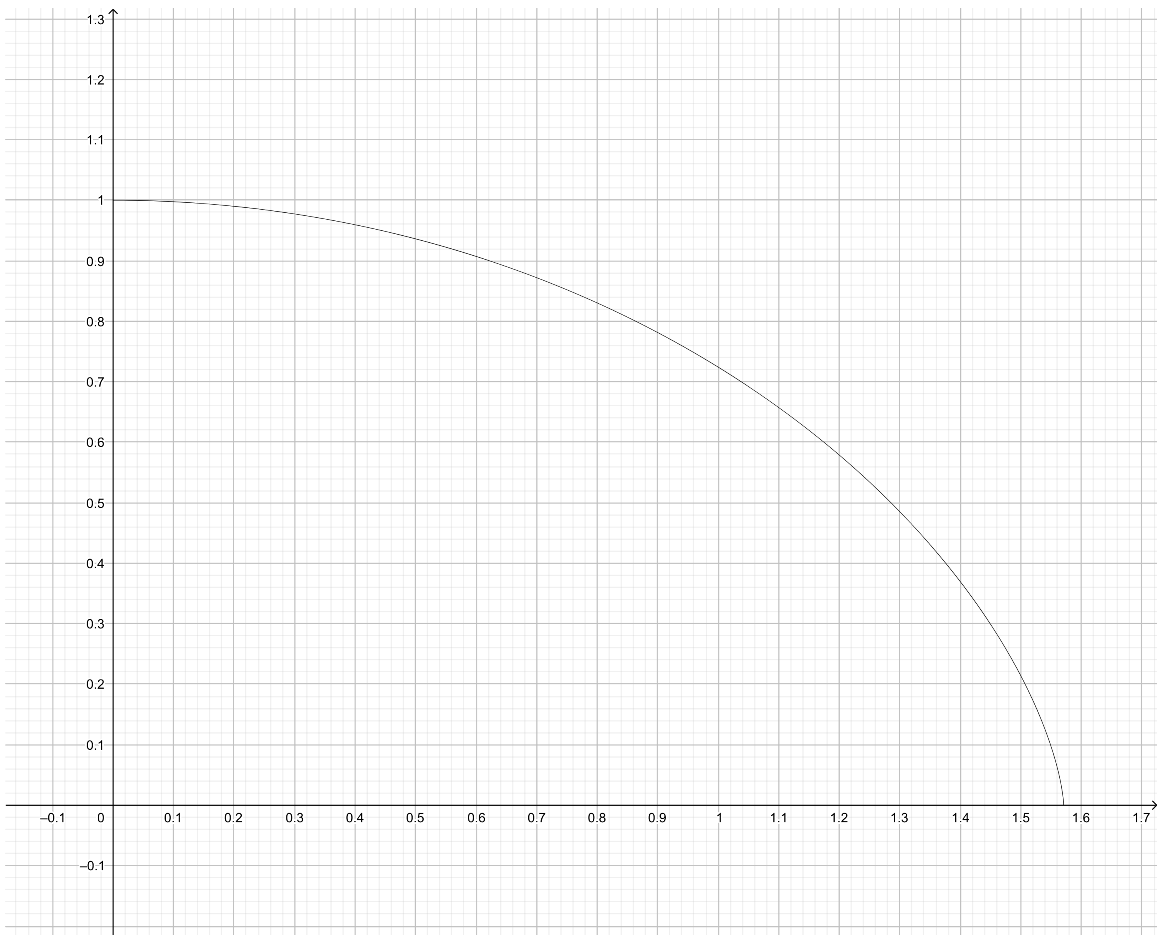

Figure 1: The graph of the parent function $x(t)$ (in other words, $k = \alpha = 1$). The graph was obtained by reflecting the graph of the parent function of $t(x)$ about the line $t = x$.