Gauss' Law

Andrew Paul

8/21/2018

In this paper, we will explore Gauss' Law. And by explore, I mean review what I have learned in Chapter 23 of H&R. I will solve some problems at the end. We begin by defining the areal vector.

Areal Vector: For a surface with zero curvature, we define the areal vector to be the vector normal to the surface with a magnitude that is the area of the surface.

For instance, consider a surface embedded in $\mathbb{R}^3$ defined by:

$$z = 0 < x, y < 1$$

This is a unit square, hence the area is 1. The areal vector of the surface is simply the standard basis vector $\mathbf{k}$. Next, we define flux.

Flux: For a vector field $\vec{F}$, and a closed surface S, the flux of $\vec{F}$ through S is defined as:

$$\oint_S \vec{F} \cdot \mathrm{d}\vec{A}$$

Electric flux is then defined as:

$$\Phi = \oint_S \vec{E} \cdot \mathrm{d}\vec{A}$$

Gauss' Law then states:

$$\boxed{\varepsilon_0 \Phi = \sum_V q}$$

Where V is the space enclosed by S, and $\varepsilon_0 \approx 8.854187817 \cdot 10^{12}\ \mathrm{C} \cdot \mathrm{V}^{-1} \cdot \mathrm{m}^{-1}$ is the permittivity of free space.

It is evident that Gauss' Law is only feasible for manual calculation when there is a great deal of symmetry to exploit. In the real world, not all Gaussian surfaces are nice and symmetrical. We will demonstrate an important corollary of Gauss' Law. That is, *the net electric flux through a Gaussian surface is not influenced by external charges.* Since I want to demonstrate my understanding and not my ability to regurgitate a textbook, I will prove this result for a sphere instead of a cylinder.

Consider a spherical surface S with center O and a point charge A outside the sphere. The electric field emanates radially outwards from A. Hence, there must exist a cone of tangency between S and the electric field of A. Let the subset of S enclosed by this cone be G. Observe that the electric field lines that enter through G must exit through S, hence the net flux due to A is 0. By the principle of superposition, all external electric fields cause a net flux of 0 through S.

Speaking of spheres, we can use Gauss' Law to derive Coulomb's Law. Consider a spherical Gaussian surface surrounding a point charge at a distance of r. The electric flux is then:

$$\Phi = \oint_S \vec{E} \cdot \mathrm{d}\vec{A} = 4\pi r^2 E$$

Which is made possible by the fact that the electric field is constant for a point charge. By Gauss' Law:

$$4\varepsilon_0 \pi r^2 E = q \Rightarrow E = \frac{1}{4\pi\varepsilon_0} \cdot \frac{q}{r^2}$$

And Coulomb's Law follows from the definition of the electric field.

We will now relate the charge density to the electric field of a Gaussian surface above a conductor with an uneven charge distribution. Consider a conductor with charge density σ, with an external electric field $\vec{E}$. That is:

$$\sigma = \frac{q}{A}$$

Expressing enclosed charge as σA, and using Gauss' law, we find that:

$$E = \frac{\sigma}{\varepsilon_0}$$

A similar idea applies to a nonconducting sheet, except unlike a conductor, the electric field may emanate in both directions orthogonal to the plane of the sheet. This is not possible in a conductor because the electric field inside a conductor is 0. As the flux is doubled, the electric field about a nonconducting sheet is given by:

$$E = \frac{\sigma}{2\varepsilon_0}$$

The shell theorems are a direct consequence of Gauss' Law.

Theorem: *The external electric field about a shell of uniform charge is equivalent to the electric field about a point charge at the center of the shell with the same charge as the shell.*

Theorem: *The net electric field inside a shell of uniform charge is zero.*

These theorems also apply to gravitational fields since Newton's Law of Universal Gravitation takes the same form as Coulomb's Law. Now for some problems.

Problem: Two nonconducting spherical shells fixed in place are on the x-axis. Shell 1 is centered at the origin, has uniform surface charge density 4.0 μC/m^2, and radius 0.50 cm. Shell 2 is centered at $(6, 0)$, has uniform surface charge density -2.0 μC/m^2, and radius 2.0 cm. Other than at $x = \infty$, where on the x-axis is the net electric field equal to zero?

Solution: By the shell theorem, we can model the external electric fields of the shells by substituting them with point charges instead. Therefore, we compute the net charges of the shells. The first shell has a net charge of:

$$q_1 = 4\pi \left(\frac{0.5}{100}\right)^2 \cdot 4 \ \mu\text{C} = \frac{\pi}{100^2} \ \mu\text{C}$$

And the second shell has a charge of:

$$q_2 = 4\pi \left(\frac{2}{100}\right)^2 \cdot -2 \ \mu\text{C} = -\frac{8\pi}{100^2} \ \mu\text{C}$$

Suppose our desired coordinate is $(x, 0)$. The distance to the first point charge is then $|x|$ and the distance to the second point charge is $|x - 6|$. The magnitudes of the electric fields from both point charges must be equal, so using the definition of the electric field from Coulomb's law:

$$\frac{k\frac{4\pi}{100^2}}{x^2} = \frac{8k\frac{4\pi}{100^2}}{(x-6)^2}$$

Solving this equation yields $\boxed{x = \dfrac{6}{1 - \sqrt{8}}}$.

Problem: Two congruent solid spheres are externally tangential to each other with radius R. Point P is the midpoint of the center of sphere 1 and the point of tangency. If the net electric field at P is zero, what is $\frac{q_2}{q_1}$?

Solution: Applying the shell theorems, we deduce that the only electric fields acting at P are from inside the shell centered at sphere 1 with radius $\frac{R}{2}$ and shell 2. We can model the electric field outside shell 2 by replacing it with a point charge at its center. The net charge of shell 2 is:

$$q_2 = \frac{4}{3}\pi\rho_2 R^3$$

Where ρ_2 is the volume density of the charge in shell 2. This produces an electric field given by:

$$E = \frac{1}{4\pi\varepsilon_0} \cdot \frac{q}{r^2}$$

Here, we have $r = \frac{3}{2}R$, as this is the distance from the center of shell 2 and P. Hence:

$$E_2 = \frac{1}{4\pi\varepsilon_0} \cdot \frac{\frac{4}{3}\pi\rho_2 R^3}{\left(\frac{3}{2}R\right)^2} = \frac{4\rho_2 R}{27\varepsilon_0}$$

Now we find the charge of the section of shell 1 bounded by a spherical surface centered at the center of shell 1 with radius $\frac{R}{2}$. The charge is:

$$q_1 = \frac{4}{3}\pi\rho_1\frac{R^3}{8} = \frac{1}{6}\pi\rho_1 R^3$$

. The electric field from this section is:

$$E_1 = \frac{1}{4\pi\varepsilon_0} \cdot \frac{\frac{1}{6}\pi\rho_1 R^3}{\left(\frac{1}{2}R\right)^2} = \frac{\rho_1 R}{6\varepsilon_0}$$

Since the net electric field at P is zero, we equate the two electric fields:

$$\frac{\rho_1 R}{6\varepsilon_0} = \frac{4\rho_2 R}{27\varepsilon_0}$$

Now observe that $q \propto \rho$, hence it suffices to find $\frac{\rho_2}{\rho_1}$. Rearranging the equation above yields:

$$\boxed{\frac{q_2}{q_1} = \frac{\rho_2}{\rho_1} = \frac{9}{8}}$$

Problem: A charge distribution that is spherically symmetric but not uniform radially produces an electric field of magnitude $E = Kr^4$, directed radially outward from the center of the sphere. Here r is the radial distance from that center, and K is a constant. What is the volume density ρ of the charge distribution?

Solution: For a spherical charge distribution, we have:

$$E = \frac{1}{4\pi\varepsilon_0} \cdot \frac{q}{r^2}$$

We are given that this is equivalent to Kr^4. Solving the resulting equation for q yields:

$$q = 4\pi K\varepsilon_0 r^6$$

Another way we can express q is by integrating the radial density of charge, $\frac{\mathrm{d}q}{\mathrm{d}r}$, from the center of the charge distribution to a desired radial distance r. Note that the radial density of charge can be expressed in terms of the volume density of charge:

$$\frac{\mathrm{d}q}{\mathrm{d}r} = \frac{\mathrm{d}q}{\mathrm{d}V} \cdot \frac{\mathrm{d}V}{\mathrm{d}r} = \rho(r) \cdot \frac{\mathrm{d}V}{\mathrm{d}r}$$

To obtain $\frac{\mathrm{d}V}{\mathrm{d}r}$, we note that for a sphere:

$$V = \frac{4}{3}\pi r^3$$

Whereupon differentiating this with respect to r reveals that $\frac{\mathrm{d}V}{\mathrm{d}r} = 4\pi r^2$. Hence:

$$\frac{\mathrm{d}q}{\mathrm{d}r} = 4\pi r^2 \rho(r)$$

Integrating this from 0 to r yields the charge:

$$q = \int \mathrm{d}q = 4\pi \int_0^r r^2 \rho(r)\ \mathrm{d}r$$

Setting this equal to our original equation for q yields:

$$K\varepsilon_0 r^6 = \int_0^r r^2 \rho(r)\ \mathrm{d}r$$

Differentiating both sides with respect to r and solving for $\rho(r)$, we find:

$$\boxed{\rho(r) = 6K\varepsilon_0 r^3}$$