Corrections to the Pendulum

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In high school physics, we are taught that a pendulum (closely) approximates simple harmonic motion. The smaller the amplitude of the oscillation the closer it is the simple harmonic motion. Why is this the case?

Recall that in *Inertial Frames of Reference and Motion in Polar Coordinates* we derived that the force on a particle in the angular direction (in polar coordinates) is

$$F_\theta = m\left(2\dot{r}\dot{\theta} + r\ddot{\theta}\right).$$

Since the length of the string (ℓ) is constant, $\dot{r} = 0$ and the Coriolis term vanishes. Therefore, we have $F = m\ell\ddot{\theta}$, which is the more common $mr\alpha$ angular form of force encountered in mechanics when the distance from the origin does not change.

We split gravity into its components to find the angular component of the net force.

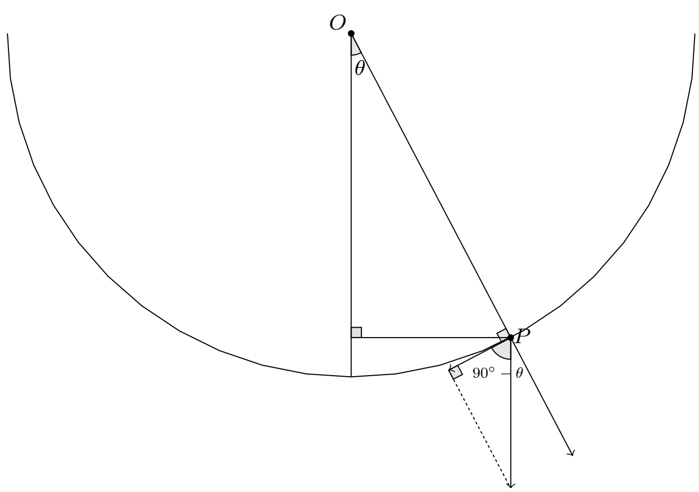

Figure 1: This diagram shows that the tangential force is $mg\cos(90° - \theta) = mg\sin\theta$.

Observe that this force is a restoring force, so its sign must be negative. Hence:

$$m\ell\ddot{\theta} = -mg\sin\theta.$$

Since $\lim_{\theta\to 0}\frac{\sin\theta}{\theta}=1$, we may approximate $\sin\theta$ with θ for small angles to obtain

$$\ddot{\theta}+\frac{g}{\ell}\theta\approx 0.$$

The solution to the differential equation when the approximation is replaced with equality is the well-known sinusoidal function, derived in *A Second Order ODE*. This is why we say that a pendulum *approximates* simple harmonic motion. Solving this approximation, we would find that the pendulum's period is given by $T=2\pi\sqrt{\frac{\ell}{g}}$.

Let us correct this result by finding the *true* period of a pendulum. This is an exercise in chapter three of *Introductory Classical Mechanics*. By our previous discussion, the correct differential equation is

$$\ddot{\theta}+\frac{g}{\ell}\sin\theta=0.$$

We use the same technique we used in *A Second Order ODE*, using the equality $\ddot{\theta}=\dot{\theta}\frac{\mathrm{d}\dot{\theta}}{\mathrm{d}\theta}$ to separate variables:

$$\begin{aligned}\int\dot{\theta}\ \mathrm{d}\dot{\theta}&=\frac{g}{\ell}\int\sin\theta\ \mathrm{d}\theta\\ \Rightarrow\dot{\theta}&=\sqrt{\frac{2g}{\ell}\cos\theta+C}.\end{aligned}$$

Suppose the maximum angle attained by the pendulum is θ_0. Then, we can use the initial condition $\dot{\theta}\big|_{\theta_0}=0$ to obtain the constant of integration $C=-\frac{2g}{\ell}\cos\theta_0$. Hence,

$$\dot{\theta}=\sqrt{\frac{2g}{\ell}(\cos\theta-\cos\theta_0)}.$$

We separate variables once again to obtain

$$\frac{\mathrm{d}\theta}{\sqrt{\cos\theta-\cos\theta_0}}=\sqrt{\frac{2g}{\ell}}\ \mathrm{d}t.$$

Now if we integrate both sides over corresponding intervals, we can preserve the equality. To find the correct interval, the bounds of integration must match up and also be useful for us (i.e. we must choose $t_0,t_1\ni t_1-t_0=T$ for the RHS and $\theta(t_0),\theta(t_1)$ for the LHS). By symmetry, the time it takes for the pendulum to swing a full period from θ_0 and back is twice the amount of time it takes for it to swing from θ_0 to $-\theta_0$ and hence four times the amount of time it takes for it to swing from 0 to θ_0. Hence,

$$\begin{aligned}8\int_0^{\theta_0}\frac{\mathrm{d}\theta}{\sqrt{\cos\theta-\cos\theta_0}}&=T\sqrt{\frac{2g}{\ell}}\\ \Rightarrow T&=\boxed{\sqrt{\frac{8\ell}{g}}\int_0^{\theta_0}\frac{\mathrm{d}\theta}{\sqrt{\cos\theta-\cos\theta_0}}}.\end{aligned}$$

The next part of the exercise gives the following instructions to find a series approximation of T in terms of θ_0.

"*It's more convenient to deal with quantities that go to* 0 *as* $\theta \to 0$*, so make use of the identity* $\cos\phi = 1 - 2\sin^2\frac{\phi}{2}$ *to write* T *in terms of sines. Then make the change of variables,* $\sin x \equiv \frac{\sin(\theta/2)}{\sin(\theta_0/2)}$. *Finally, expand your integrand judiciously in powers of (the fairly small quantity)* θ_0*, and perform the integrals to show...*" [approximation redacted for suspense purposes].

We follow the steps. For now, we only worry about the integral, I, disregarding the constant of $\sqrt{\frac{8\ell}{g}}$. Let the integrand of I be G. We use the provided half-angle identity to rewrite G as

$$\begin{aligned} G &= \frac{1}{\sqrt{\cos\theta - \cos\theta_0}} \\ &= \frac{1}{\sqrt{1 - 2\sin^2\frac{\theta}{2} - 1 + 2\sin^2\frac{\theta_0}{2}}} \\ &= \frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{\sin^2\frac{\theta_0}{2} - \sin^2\frac{\theta}{2}}}. \end{aligned}$$

Let us amend G and I to exclude the pesky $\frac{1}{\sqrt{2}}$ and absorb this in the existing constant to make it $2\sqrt{\frac{\ell}{g}}$.

Next, we use the substitution provided. For the rest of the solution, let $k = \sin\frac{\theta_0}{2}$. Then, implicit differentiation on $\sin x = \frac{\sin(\theta/2)}{\sin(\theta_0/2)}$ provides

$$\mathrm{d}\theta = \frac{2k\cos x}{\cos\frac{\theta}{2}}\,\mathrm{d}x = \frac{2k\cos x}{\sqrt{1 - k^2\sin^2 x}}\,\mathrm{d}x.$$

Now we're ready to perform the substitution on G.

$$\begin{aligned} G &= \frac{1}{\sqrt{k^2 - \sin^2\frac{\theta}{2}}} \\ &= \frac{1}{\sqrt{k^2 - k^2\sin^2\theta}} \\ &= \frac{1}{k\cos x}. \end{aligned}$$

We put this together with $\mathrm{d}\theta$ to write I, discarding the extra factor of 2 and absorbing it into the existing constant to make it $4\sqrt{\frac{\ell}{g}}$.

$$I = \int_0^{\frac{\pi}{2}} \frac{\mathrm{d}x}{\sqrt{1 - k^2\sin^2 x}}.$$

Observe that the bounds of integration have also changed due to our substitution. This particular integral with those particular bounds is known as the *complete elliptic integral of the first kind*. Its name derives from its historical origins as it was first studied to find the arc length of an ellipse. There is no solution in elementary functions, which is why we did not bother actually tackling the integral when we found the exact expression for T.

Nonetheless, it is perfectly possible to obtain an infinite series for I. We continue by performing a binomial expansion on G. This can be done with the generalized binomial formula, or simply with the binomial series, which is the Taylor series of $(1+y)^z$:

$$(1+y)^z = \sum_{r=0}^{\infty} \binom{z}{r} y^r,$$

where $\binom{z}{y}$ is the *generalized binomial coefficient* which we have already encountered in *The Vandermonde Convolution*. It is defined by

$$\binom{z}{r} = \frac{1}{r!} \prod_{j=0}^{r-1} (z-j).$$

The convergence of the binomial series can be found with the ratio test. We must satisfy:

$$\left| y \lim_{r\to\infty} \frac{z-r}{r+1} \right| < 1,$$

or $|y| < 1$ for the series to converge absolutely. This is obviously the case because $0 < k < 1$ and $0 \le \sin x \le 1$ and so $0 < |y| = k^2 \sin^2 x < 1$.

Substituting $y = -k^2 \sin^2 x$ and $z = -\frac{1}{2}$, we have

$$\begin{aligned} G &= \sum_{r=0}^{\infty} (-1)^r \binom{-1/2}{r} (k \sin x)^{2r} \\ &= 1 + \frac{1}{2} k^2 \sin^2 x + \frac{3}{8} k^4 \sin^4 x + \frac{5}{16} k^6 \sin^6 x + \frac{35}{128} k^8 \sin^8 x + \frac{63}{256} k^{10} \sin^{10} x + ... \end{aligned}$$

There are a lot of powers of sines, so naturally we derive the sine reduction formula. We wish to reduce $\int \sin^n x \ \mathrm{d}x$. Letting $u = \sin^{n-1} x$ and $\mathrm{d}v = \sin x \ \mathrm{d}x$, we have $\mathrm{d}u = (n-1)\sin^{n-2} x \cos x \ \mathrm{d}x$ and $v = -\cos x$. Integrating by parts,

$$\begin{aligned} \int \sin^n x \ \mathrm{d}x &= -\sin^{n-1} x \cos x + (n-1) \int \sin^{n-2} x \cos^2 x \ \mathrm{d}x \\ &= -\sin^{n-1} x \cos x + (n-1) \int \sin^{n-2} x (1 - \sin^2 x) \ \mathrm{d}x \\ &= -\sin^{n-1} x \cos x + (n-1) \int \sin^{n-2} x \ \mathrm{d}x - (n-1) \int \sin^n x \ \mathrm{d}x \\ &= -\frac{1}{n} \sin^{n-1} x \cos x + \frac{n-1}{n} \int \sin^{n-2} x \ \mathrm{d}x, \end{aligned}$$

where in the last step we simply rearranged the equation to solve for the desired integral.

When we integrate G term-by-term to compute the definite integral, I, we evaluate each term's antiderivative at the bounds of integration, subtract the evaluations, and then sum the results across all the terms, by the linearity of integration and the fundamental theorem of calculus. Note that every term in G has an even exponent of sine, so it suffices to evaluate the antiderivative of $\sin^n x$ at $\frac{\pi}{2}$ and 0 for even n only.

Observe that due to our reduction formula, every term in the antiderivative of $\sin^n x$ has a power of sine in it, excluding the last term. Since n is even, the last term is $\propto \int \sin^0 x \, \mathrm{d}x = x$. But this term, along with all the other terms with powers of sine, will vanish when evaluated at $x = 0$. Hence, the antiderivative at evaluated at 0 is just 0.

Next, we evaluate the antiderivative at $\frac{\pi}{2}$. This time, while most of the terms vanish due to their cosines, the last term does not. We find that this last term is:

$$\frac{n-1}{n}\left(0+\frac{n-3}{n-2}\left(0+\frac{n-5}{n-4}\left(0+...+\left(\frac{1}{2}\cdot\frac{\pi}{2}\right)...\right)\right)\right)=\frac{\pi}{2}\prod_{i=0}^{\frac{n}{2}-1}\frac{n-(2i+1)}{n-2i}.$$

Now we can write down I using term-by-term integration on G:

$$\begin{aligned} I &= \sum_{r=0}^{\infty}\int_0^{\frac{\pi}{2}}(-1)^r\binom{-1/2}{r}(k\sin x)^{2r}\,\mathrm{d}x \\ &= \frac{\pi}{2}\left[1+\sum_{r=1}^{\infty}(-1)^r\binom{-1/2}{r}k^{2r}\prod_{i=0}^{r-1}\frac{2r-(2i+1)}{2r-2i}\right]. \end{aligned}$$

We now have our series representation of T:

$$\boxed{T = 2\pi\sqrt{\frac{\ell}{g}}\left[1+\sum_{r=1}^{\infty}(-1)^r\binom{-1/2}{r}\left(\sin^{2r}\frac{\theta_0}{2}\right)\prod_{i=0}^{r-1}\frac{2r-(2i+1)}{2r-2i}\right]}.$$

Observe that it $\lim_{\theta_0\to 0} T = 2\pi\sqrt{\frac{\ell}{g}}$ as expected.

We finish by finding the desired approximation. Due to the limit $\lim_{\theta\to 0}\frac{\sin(\theta/2)}{\theta} = \frac{1}{2}$, we can approximate $\sin(\theta/2) \approx \frac{\theta_0}{2}$ for small angles θ_0. We can then approximate T as

$$\begin{aligned} T &\approx 2\pi\sqrt{\frac{\ell}{g}}\left[1+\sum_{r=1}^{\infty}(-1)^r\left(\frac{\theta_0}{2}\right)^{2r}\binom{-1/2}{r}\prod_{i=0}^{r-1}\frac{2r-(2i+1)}{2r-2i}\right] \\ &= 2\pi\sqrt{\frac{\ell}{g}}\left(1+\frac{1}{16}\theta_0^2+\frac{9}{1024}\theta_0^4+\frac{25}{16384}\theta_0^6+...\right). \end{aligned}$$

But can we do better? Can we find an *exact* power series for T?

It is possible to use the Taylor series of $\sin^{2r}\frac{\theta_0}{2}$ and carefully combine terms to find an *exact* infinite series for T. The problem is that there is no easy way to express the Taylor expansion of

$\sin^{2r}\frac{\theta_0}{2}$ explicitly as a function of r. We can make a couple observations to try to write this as succinctly as possible

First, we observe that the term of lowest degree in the expansion of $\sin^{2r}\frac{\theta_0}{2}$ has degree $2r$. This follows from the fact that the Taylor series of sine has a linear term of lowest degree, which means that when we raise that series to the $2r$ power, the resulting lowest degree term will have degree $2r$.

This means that powers of θ_0 with degree $2r_0$ appear in the terms corresponding with when $r = 1$ through $\frac{r_0}{2}$ in the series for T. We can see this by writing T as:

$$\begin{aligned}
T &= 2\pi\sqrt{\frac{\ell}{g}}\left[1 + \frac{1}{4}\sin^2\frac{\theta_0}{2} + \frac{9}{64}\sin^4\frac{\theta_0}{2} + \frac{25}{256}\sin^6\frac{\theta_0}{2} + ...\right] \\
&= 2\pi\sqrt{\frac{\ell}{g}}\left[1 + \frac{1}{4}\left(\frac{\theta_0^2}{4} - \frac{\theta_0^4}{48} + \frac{\theta_0^6}{1440} - ...\right) + \frac{9}{64}\left(\frac{\theta_0^4}{16} - \frac{\theta_0^6}{96} + \frac{\theta_0^8}{1280} - ...\right) + \frac{25}{256}\left(\frac{\theta_0^6}{64} - \frac{\theta_0^8}{256} + ...\right) + ...\right] \\
&= 2\pi\sqrt{\frac{\ell}{g}}\left[1 + \frac{1}{16}\theta_0^2 + \left(\frac{9}{1024} - \frac{1}{192}\right)\theta_0^4 + \left(\frac{25}{16384} - \frac{119}{92160}\right)\theta_0^6 + ...\right] \\
&= \boxed{2\pi\sqrt{\frac{\ell}{g}}\left[1 + \frac{1}{16}\theta_0^2 + \frac{11}{3072}\theta_0^4 + \frac{173}{737280}\theta_0^6 + ...\right]}.
\end{aligned}$$

In the third line, the errors caused by using the approximation $\sin(\theta_0/2) \approx \frac{\theta_0}{2}$ on the higher order terms are shown. In particular, the coefficient of the fourth-order term was off by $\frac{1}{192}$ and the coefficient of the sixth-order term was off by $\frac{119}{92160}$. These are relatively large corrections, though their effects are only really observable when θ_0 becomes large enough that its fourth, sixth, and higher powers become non-negligible.

Putting this all together, we conclude

$$\boxed{T = 2\pi\sqrt{\frac{\ell}{g}}\left[1 + \sum_{n=2}^{\infty}\sum_{r=1}^{\frac{r}{2}}(-1)^r\binom{-1/2}{r}C_{2r}(n)\theta_0^n\prod_{i=0}^{r-1}\frac{2r-(2i+1)}{2r-2i}\right]},$$

where $C_{2r}(n)$ is the coefficient of the n^{th} order term in the Taylor expansion of $\sin^{2r}\frac{\theta_0}{2}$.