Minimal Height of the Center of Mass of a Uniform Cylinder filled with Fluid

Andrew Paul

7/14/2018

In this analysis, we consider an empty cylinder of radius r, height h, and uniformly distributed mass m_c. We will be filling the cylinder with a fluid with density ρ up to a level l. We suppose that the cylinder is symmetric about the y-axis. Naturally, the y-coordinate of the center of mass is given by:

$$y_c = \frac{1}{M} \int y \ \mathrm{d}m$$

Our cylinder is split into two sections, namely the fluid-filled section and the empty section. We must first find the linear mass densities ($\lambda = \frac{\mathrm{d}m}{\mathrm{d}y}$) of both. As we are assuming uniform distributions, the λ values of both sections will be constants. The linear density of the empty section is that of the entire cylinder when empty, which is:

$$\lambda_c = \frac{m_c}{h}$$

On the other hand, the linear density of the filled section is given by:

$$\lambda_f = \frac{m_{c \text{ f section}} + m_{f \text{ f section}}}{l} = \frac{\lambda_c l + \pi \rho r^2 l}{l} = \lambda_c + \pi \rho r^2$$

It is evident that since our linear density is piecewise, so too must be our integral. Next, we find the total mass of the system, which is fairly straightforward:

$$M = m_c + \pi \rho r^2 l$$

Under substitution, our integral becomes:

$$y_c = \frac{\lambda}{M} \int y \ \mathrm{d}y$$

And in our case this is:

$$\frac{\lambda_f}{M} \int_0^l y \ \mathrm{d}y + \frac{\lambda_c}{M} \int_l^h y \ \mathrm{d}y$$

We compute this:

$$\begin{aligned}\frac{\frac{\lambda_f}{2}l^2 + \frac{\lambda_c}{2}h^2 - \frac{\lambda_c}{2}l^2}{M} &= \frac{\lambda_f l^2 + \lambda_c h^2 - \lambda_c l^2}{2m_c + 2\pi\rho r^2 l} \\ &= \frac{(\lambda_c + \pi\rho r^2)l^2 + \lambda_c h^2 - \lambda_c l^2}{2m_c + 2\pi\rho r^2 l} \\ &= \boxed{\frac{m_c h + \pi\rho r^2 l^2}{2m_c + 2\pi\rho r^2 l}}\end{aligned}$$

As a check, we observe that letting $l = h$ correctly predicts that $y_c = \frac{h}{2}$. In other words, a uniform cylinder completely filled with fluid has a center of mass at the center of the cylinder.

But what configuration is the most stable? In other words, how much fluid would we need to add in order to minimize the height of the center of mass? We calculate this by expressing the above expression for y_c as a function of l:

$$y_c(l) = \frac{m_c h + \pi\rho r^2 l^2}{2m_c + 2\pi\rho r^2 l}$$

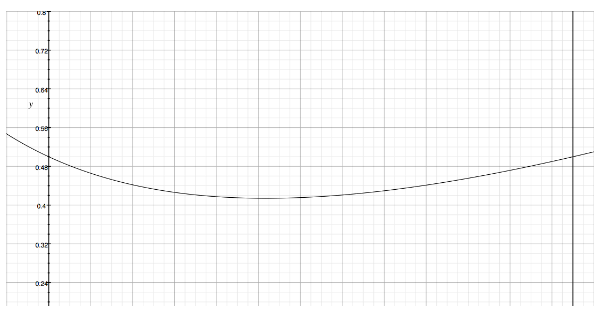

Figure 1: The graph of the parent function $f(x) = \frac{1+x^2}{2+2x}$ within the relevant bounds of $x = 0$ and $x = h = 1$.

Taking the derivative:

$$\frac{\mathrm{d}y_c}{\mathrm{d}l} = \frac{1}{2}\left(\frac{2\pi\rho r^2 l(m_c + \pi\rho r^2 l) - \pi\rho r^2(m_c h + \pi\rho r^2 l^2)}{(m_c + \pi\rho r^2 l)^2}\right)$$

Setting all of this equal to 0 yields:

$$\pi\rho r^2 l^2 + 2m_c l - m_c h = 0$$

Solving this quadratic, we have the solutions:

$$l = \frac{-m_c \pm \sqrt{m_c^2 + \pi\rho r^2 m_c h}}{\pi\rho r^2}$$

But since $\sqrt{m_c^2 + \pi\rho r^2 m_c h} > m_c$, we conclude that the only solution is:

$$\boxed{l_{\text{min}} = \frac{-m_c + \sqrt{m_c^2 + \pi\rho r^2 m_c h}}{\pi\rho r^2}}$$

We will now derive an interesting result: when the height of the center of mass is minimized, it coincides with the fluid level. Intuitively, this makes sense. The moment the fluid level begins to exceed the height of the center of mass, it will start to pull the center of mass upwards. To establish this result, we substitute our above expression for l_{min} into $y_c(l)$:

$$\begin{aligned}
y_c(l_{\text{min}}) &= \frac{m_c h + \pi\rho r^2 \left(\frac{-m_c + \sqrt{m_c^2 + \pi\rho r^2 m_c h}}{\pi\rho r^2}\right)^2}{2m_c + 2\pi\rho r^2 \left(\frac{-m_c + \sqrt{m_c^2 + \pi\rho r^2 m_c h}}{\pi\rho r^2}\right)} \\
&= \frac{1}{2} \cdot \frac{m_c h + \frac{1}{\pi\rho r^2}\left(2m_c^2 + \pi\rho r^2 m_c h - 2m_c\sqrt{m_c^2 + \pi\rho r^2 m_c h}\right)}{\sqrt{m_c^2 + \pi\rho r^2 m_c h}} \\
&= \frac{2}{2} \cdot \frac{m_c h + \frac{m_c^2}{\pi\rho r^2} - \frac{m_c\sqrt{m_c^2 + \pi\rho r^2 m_c h}}{\pi\rho r^2}}{\sqrt{m_c^2 + \pi\rho r^2 m_c h}} \\
&= \frac{m_c h + \frac{m_c^2}{\pi\rho r^2}}{\sqrt{m_c^2 + \pi\rho r^2 m_c h}} - \frac{m_c}{\pi\rho r^2} \\
&= \frac{\left(\pi\rho r^2 m_c h + m_c^2\right)\sqrt{m_c^2 + \pi\rho r^2 m_c h}}{\pi\rho r^2\left(m_c^2 + \pi\rho r^2 m_c h\right)} - \frac{m_c}{\pi\rho r^2} \\
&= \frac{-m_c + \sqrt{m_c^2 + \pi\rho r^2 m_c h}}{\pi\rho r^2} \\
&= \boxed{l_{\text{min}}}
\end{aligned}$$

Hence, the fluid level must coincide with the height of the center of mass when the latter is minimized.

Is this result beautiful? After all, it *is* intuitively obvious. But at the heart of beauty is simplicity. And I'd say that the madness above yielding a surprisingly simple result qualifies this as a gem.