Problem (Canada 2001/1):

Randy: "Hi Rachel that's an interesting quadratic equation you have written down. What are its roots?"
Rachel: "The roots are two positive integers. One of the roots is my age, and the other root is the age of my younger brother, Jimmy."
Randy: "That is very neat! Let me see if I can figure out how old you and Jimmy are. That shouldn't be too difficult since all of your coefficients are integers. By the way, I notice that the sum of the three coefficients is a prime number."
Rachel: "Interesting. Now figure out how old I am."
Randy: "Instead, I will guess your age and substitute it for x in your quadratic equation...darn, that gives me -55, and not 0."
Rachel: "Oh, leave me alone!"

Determine the ages of Rachel and Jimmy.

Solution (Andrew Paul): We write our polynomial in its factored form:

$$R(x) = ax^2 + bx + c = a(x - r)(x - j)$$

Where r and j are the ages of Rachel and Jimmy respectively. Note that letting $x = 1$ yields the sum of the roots $a + b + c$ which we will denote as prime number p:

$$p = a(1 - r)(1 - j)$$

Observe that the product of two of the three factors on the RHS must be 1 to avoid contradicting the primality of p. We cannot have $1 - r = 1 - j = \pm 1$ because this implies $r = j$ which is not true (Rachel is older). Furthermore, $1 - r$ and $1 - j$ cannot be reciprocals because $r, j \in \mathbb{Z}$ and a cannot be the reciprocal of either $1 - r$ or $1 - j$ because $a \in \mathbb{Z}$. Hence, two of the three factors must be ± 1, one of which is a.

Immediately, this yields $1 - j = -1 \Rightarrow j = 2$ as the only valid solution for $j > 0$ (Rachel must be older than 2 so we let Jimmy be 2). Now note that $1 - r < 0$ for $r > 1$. Clearly, Rachel's age is greater than 1 so this implies that this factor will be negative (in particular, it must equal $-p$). Hence:

$$p = ap \Rightarrow a = 1$$

We've determined three important things so far. We know $a = 1$, $j = 2$ and that r is one more than a prime. If we agree that three-year-olds are unlikely to be proficient with the English language, quadratic equations, and number theory, then this prime cannot be 2 so we can deduce that Rachel's age is an even number. It is also known that for some $k \in \mathbb{Z}$:

$$(k - 2)(k - r) = -55$$

Splitting -55 into its integral factors and equating them to the factors on the LHS gives us systems of equations. We find that the only factors that work are $k - 2 = 5 \Rightarrow k = 7$ and $k - r = 7 - r = -11 \Rightarrow r = 18$.

Hence Rachel is 18 years old and Jimmy is 2 years old. □

Remark: Even though $j = 2$ follows along with $a = 1$, if we had assumed R was monic, we could've deduced $j = 2$ and then continued to construct a valid R without analyzing the factorization of R as follows:

$$R(x) = x^2 + bx + c$$

It is given that $b + c + 1 = p$ for some prime p. This means that p is instead some odd prime. So by Vieta, we have:

$$rj - (r + j) = p - 1$$

We complete the rectangle and factor:

$$(r - 1)(j - 1) = p$$

Thus $j - 1 = 1 \Rightarrow j = 2$ and r is one greater than a prime as expected. The rest of the solution follows as before, and we add the additional step of checking our final polynomial R to ensure that it satisfies the constraints since we assumed that it was monic before proceeding.