Problem (2017 FAMAT Fall Interschool/23): Define $f_1(x) = \sin\frac{\pi}{2}x$, $f_2(x) = \sin\pi x$, $f_3(x) = \cos\frac{\pi}{2}x$, and $f_4(x) = \cos\pi x$. For some positive integer n we have many possible sequences $\{a_i \in \{1,2,3,4\}\}_{i=1}^{n}$. For example, a possible sequence for $n = 2$ is $\{4, 1\}$. Then for each sequence, define a unique function:

$$G_{\{a_i\}}(x) = f_{a_1}(f_{a_2}(f_{a_3}(...f_{a_n}(x)...)))$$

Given that each valid sequence $\{a_i\}$ is equally possible, and the expected value E of $\frac{\mathrm{d}}{\mathrm{d}x}G_{\{a_i\}}(x)\Big|_{x=0}$ over all these sequences satisfies $\left\lfloor \frac{E}{2017} \right\rfloor = 1$, find the least positive integer value of n.

Solution (Andrew Paul): The expected value E is the dot product of the n-dimensional vectors $\mathbf{V}$ and $\mathbf{P}$ where each component of $\mathbf{V}$ is one of the n values of $G_{\{a_i\}}(x)$ (not all n values are necessarily unique) and each component of $\mathbf{P}$ is the corresponding probability of each component in $\mathbf{V}$. Since every sequence is equally possible, the components of $\mathbf{P}$ are all equal:

$$\mathbf{P} = \underbrace{\langle p, p, p, ..., p\rangle}_{n \text{ components}}$$

To determine p, we simply count the number of sequences $\{a_i\}$ are possible for a given n. This is simple as we always have 4 options for each element thus the number of possible sequences is 4^n. Since each sequence is equally likely, we find that $p = \frac{1}{4^n}$.

Also observe that since each component, p, of $\mathbf{P}$ is equal, we can factor it out when computing $\mathbf{P} \cdot \mathbf{V}$. In other words, let $\mathbf{V} = \langle v_1, v_2, v_3, ..., v_n\rangle$. Then:

$$E = \mathbf{P} \cdot \mathbf{V} = pv_1 + pv_2 + pv_3 + ... + pv_n = p(v_1 + v_2 + v_3 + ... + v_n) = \frac{1}{4^n}\sum_{i=1}^{n} v_i$$

So it suffices to compute the sum of the components of $\mathbf{V}$.

Now, let us differentiate $G_{\{a_i\}}(x)$. Suppose that $r > s$ and that both are integers. Let:

$$h_{s...r} = h_s \circ h_{s+1} \circ h_{s+2} \circ ... \circ h_r$$

For some functions h_i. Let $h_{r...r} = h_r$. Then by chain rule:

$$h'_{1...r} = \left(h'_1 \circ h_{2...r}\right)\left(h'_2 \circ h_{3...r}\right) ... \left(h'_{r-1} \circ h_{r...r}\right) h'_r = \prod_{k=1}^{r}\left(h'_k \circ h_{k+1...n}\right)$$

So the derivative of $G_{\{a_i\}}(x)$ is:

$$\frac{\mathrm{d}}{\mathrm{d}x}G_{\{a_i\}}(x) = \left(f'_{a_1} \circ f_{a_2...a_n}\right)\left(f'_{a_2} \circ f_{a_3...a_n}\right) ... \left(f'_{a_{n-1}} \circ f_{a_n...a_n}\right) f'_{a_n}$$

We seek an expression in terms of n for the sum of these for a given n. For $n = 2$ we only have $4^2 = 16$ possibilities, so we list out all of the possible derivatives of $f_{a_1} \circ f_{a_2}$ at $x = 0$. We find that following are our only nonzero results:

$$\{a_i\} = \{1, 1\} \to \frac{\mathrm{d}}{\mathrm{d}x}G_{\{a_i\}}(x)\Big|_{x=0} = \frac{\pi^2}{4}$$

$$\{a_i\} = \{1, 2\} \to \frac{\mathrm{d}}{\mathrm{d}x} G_{\{a_i\}}(x)\bigg|_{x=0} = \frac{\pi^2}{2}$$

$$\{a_i\} = \{2, 1\} \to \frac{\mathrm{d}}{\mathrm{d}x} G_{\{a_i\}}(x)\bigg|_{x=0} = \frac{\pi^2}{2}$$

$$\{a_i\} = \{2, 2\} \to \frac{\mathrm{d}}{\mathrm{d}x} G_{\{a_i\}}(x)\bigg|_{x=0} = \pi^2$$

Observe that in general, allowing $a_i \in \{3, 4\}$ results in $\frac{\mathrm{d}}{\mathrm{d}x} G_{\{a_i\}}(x)\Big|_{x=0} = 0$ because we would end up taking the sine of 0 which forces the entire product to become 0. So when taking the sum of the possible values $\frac{\mathrm{d}}{\mathrm{d}x} G_{\{a_i\}}(x)\Big|_{x=0}$, we can ignore these cases. Thus, we have:

$$\sum_{i=1}^{2} v_i = \frac{\pi^2}{4} + \frac{\pi^2}{2} + \frac{\pi^2}{2} + \pi^2 = \frac{9\pi^2}{4}$$

Now we move on to $n = 3$. Keeping in mind our restriction of $a_i \in \{1, 2\}$, we look for the nonzero cases. Notice by our work with the chain rule, the value of the derivative for three composed functions will be the value of the inner two composed functions inside the derivative of the third outer function times the value of the derivative of the composition of the two inner functions.

For instance, let us take a look at the case where we have f_1 as both of our inner two functions. The third outer function must be either f_1 or f_2 (because as mentioned before, we would otherwise have a derivative of 0 at $x = 0$.) Note that the derivatives these functions are $f_1'(x) = \frac{\pi}{2} \cos \frac{\pi}{2} x$ and $f_2'(x) = \pi \cos \pi x$. Since for $a_{i,j} \in \{1, 2\}$, we must have $f_{a_i}\left(f_{a_j}(0)\right) = 0$ as both functions would be sine functions, we must have $f_{a_1}' \circ f_{a_2} \circ f_{a_3} = \frac{\pi}{2}$ or $f_{a_1}' \circ f_{a_2} \circ f_{a_3} = \pi$ depending on whether we choose $f_{a_1} = f_1$ or $f_{a_1} = f_2$ respectively.

This gives us two cases for when we have f_1 as our inner two functions. The derivative evaluated at 0 may have the value of $\frac{\pi^2}{4} \cdot \frac{\pi}{2}$ or $\frac{\pi^2}{4} \cdot \pi$. Since our goal is to sum all possible values, we consider both of these. The other cases for $n = 3$ are similar when choosing the inner two functions to be any permutation of a combination of f_1 and f_2. There will always be a case with an additional factor of $\frac{\pi}{2}$ which results from making the third outer function f_1 and there will always be a case with an additional factor of π which results from making the third outer function f_2.

Hence, the sum of the possibilities for $n = 3$ is:

$$\left(\frac{\pi}{2}\right)\left(\frac{\pi^2}{4}\right) + \left(\frac{\pi}{2}\right)\left(\frac{\pi^2}{2}\right) + \left(\frac{\pi}{2}\right)\left(\frac{\pi^2}{2}\right) + \left(\frac{\pi}{2}\right)\left(\pi^2\right) + (\pi)\left(\frac{\pi^2}{4}\right) + (\pi)\left(\frac{\pi^2}{2}\right) + (\pi)\left(\frac{\pi^2}{2}\right) + (\pi)\left(\pi^2\right)$$

Factoring, this reduces to:

$$\underbrace{\frac{\pi}{2}\left(\frac{\pi^2}{4} + \frac{\pi^2}{2} + \frac{\pi^2}{2} + \pi^2\right)}_{\text{Where } f_{a_1} = f_1} + \underbrace{\pi\left(\frac{\pi^2}{4} + \frac{\pi^2}{2} + \frac{\pi^2}{2} + \pi^2\right)}_{\text{Where } f_{a_1} = f_2} = \left(\frac{3\pi}{2}\right)\left(\frac{9\pi^2}{4}\right) = \sum_{i=1}^{3} v_i$$

Now the pattern is clear. For $n = 4$ we have:

$$\underbrace{\frac{\pi}{2}\left(\left(\frac{\pi}{2}\right)\left(\frac{9\pi^2}{4}\right) + \pi\left(\frac{9\pi^2}{4}\right)\right)}_{\text{Where } f_{a_1} = f_1} + \underbrace{\pi\left(\left(\frac{\pi}{2}\right)\left(\frac{9\pi^2}{4}\right) + \pi\left(\frac{9\pi^2}{4}\right)\right)}_{\text{Where } f_{a_1} = f_2}$$

This reduces to:

$$\left(\frac{3\pi}{2}\right)^2\left(\frac{9\pi^2}{4}\right)$$

In general, we conjecture that for $n \geq 3$:

$$\sum_{i=1}^{n} v_i = \frac{9\pi^2}{4}\left(\frac{3\pi}{2}\right)^{n-2}$$

Though this is heuristically obvious, we must still prove it using induction. It is true for our base case of $n = 3$. Now suppose it is true for any k. Then, to obtain the sum for $k+1$, we use previous observations to deduce:

$$\sum_{i=1}^{k+1} v_i = \frac{\pi}{2}\left(\frac{9\pi^2}{4}\right)\left(\frac{3\pi}{2}\right)^{k-2} + \pi\left(\frac{9\pi^2}{4}\right)\left(\frac{3\pi}{2}\right)^{k-2}$$

But this factors as:

$$\sum_{i=1}^{k+1} v_i = \frac{9\pi^2}{4}\left(\frac{3\pi}{2}\right)^{k-1}$$

Completing our induction.

We're nearly done. Recall:

$$E = \frac{1}{4^n}\sum_{i=1}^{n} v_i$$

We now have a closed form for the summation in n. Substituting for this:

$$E = \frac{9\pi^2}{4^{n+1}}\left(\frac{3\pi}{2}\right)^{n-2} = \frac{(3\pi)^2}{2^{2n+2}}\left(\frac{(3\pi)^{n-2}}{2^{n-2}}\right) = \frac{(3\pi)^n}{2^{3n}} = \left(\frac{3\pi}{8}\right)^n$$

Since $\left\lfloor \frac{E}{2017} \right\rfloor = 1$, we have $2017 \leq E < 4034$. Hence:

$$2017 \leq \left(\frac{3\pi}{8}\right)^n < 4034$$

The logarithm is a monotonically increasing function, so we may take the natural logarithm and preserve the inequality:

$$\frac{\log 2017}{\log \pi + \log 3 - \log 8} \leq n < \frac{\log 4034}{\log \pi + \log 3 - \log 8}$$

We find that the LHS is approximately 46.4 and that the RHS is approximately 50.7. Since n is an integer, we may write:

$$47 \leq n \leq 50$$

Hence $\boxed{n_{\min} = 47}$ and we are done. □