**The Bounce Equations (Andrew Paul):** Behold, the bounce equations!

$$y_{\vec{s}} = \frac{\vec{a}_g(x_{\vec{s}}+S)^2(e^{2n-2}\tan^2\theta_1+1)}{2v_i^2(e^{2n-2}\sin^2\theta_1+\cos^2\theta_1)} + e^{n-1}(x_{\vec{s}}+S)\tan\theta_1,\ S = \begin{cases} 0 \text{ For } n=1 \\ \sum_{k=1}^{n-1}\left(\frac{2v_i^2 e^{k-1}\tan\theta_1(e^{2k-2}\sin^2\theta_1+\cos^2\theta_1)}{\vec{a}_g(e^{2k-2}\tan^2\theta_1+1)}\right) \text{ For } n\geq 2 \end{cases}$$

Is this insane? Yes. Is it real? Yes.

The derivation of these equations are heavy on trigonometry. First let us try to understand the structure of the equation. The first equation gives $y_{\vec{s}}$ in terms of gravitational acceleration ($\vec{a}_g$), initial velocity of the first launch ($v_i$), launch angle of the first launch ($\theta_1$), the horizontal displacement ($x_{\vec{s}}+S$), the coefficient of restitution ($e$), and the bounce number ($n$). For a positive integer value $n$, graphing the equation will give the parabolic trajectory of the $n$th bounce *with the appropriate horizontal shift away from the origin or initial launch point.* This horizontal shift is given by $S$. Consider what happens when we let $n=1$. Since this is the initial launch with no horizontal shift away from the starting point (note $S=0$ when $n=1$), and there is no effect of the coefficient of restitution (because this is the first launch and there was no bounce against the ground before this), our equation should function in the same way as our previous parabolic trajectory equation! Trying it out we have:

$$y_{\vec{s}} = \frac{\vec{a}_g x_{\vec{s}}^2(\tan^2\theta_1+1)}{2v_i^2(\sin^2\theta_1+\cos^2\theta_1)} + x_{\vec{s}}\tan\theta_1$$

Applying the Pythagorean identities on the trigonometric functions further reduces the equation to:

$$y_{\vec{s}} = \frac{\vec{a}_g x_{\vec{s}}^2\sec^2\theta_1}{2v_i^2} + x_{\vec{s}}\tan\theta_1 = \frac{\vec{a}_g x_{\vec{s}}^2}{2v_i^2\cos^2\theta_1} + x_{\vec{s}}\tan\theta_1$$

Holy Euler! When $n=1$ our equation does indeed degenerate into the parabolic trajectory equation! The point is, the parabolic trajectory of the projectile after each bounce can still be modeled by the parabolic trajectory equation! After all, the parabolic trajectory equation describes *every possible parabolic trajectory.* The Bounce Equations extend the parabolic trajectory equation. Essentially, the trajectories are made shorter each bounce according to the coefficient of restitution and obviously the trajectories don't keep starting at the origin; they start where the previous ones left off. So deriving the Bounce Equations becomes a matter of adjusting the parabolic trajectory equation to factor in the coefficient of restitution and the horizontal shift of the trajectory from the starting point for each bounce.

Take another look at the parabolic trajectory equation:

$$y_{\vec{s}} = \frac{\vec{a}_g x_{\vec{s}}^2}{2v_i^2\cos^2\theta_1} + x_{\vec{s}}\tan\theta_1$$

The first step in generalizing it to all bounces, not just the initial launch, is changing $v_i$ to $v_{ni}$ and $\theta_1$ to $\theta_n$. This gives:

$$y_{\vec{s}} = \frac{\vec{a}_g x_{\vec{s}}^2}{2v_{ni}^2\cos^2\theta_n} + x_{\vec{s}}\tan\theta_n$$

But recall in previous papers, we derived the two following equations:

$$\theta_n = \arctan\left(e^{n-1}\tan\theta_1\right)$$

$$v_{ni} = v_i\sqrt{e^{2n-2}\sin^2\theta_1+\cos^2\theta_1}$$

Substituting these into our above generalized parabolic trajectory equation gives us:

$$y_{\vec{s}} = \frac{\vec{a}_g x_{\vec{s}}^2}{2v_i^2(e^{2n-2}\sin^2\theta_1+\cos^2\theta_1)\cos^2\left[\arctan\left(e^{n-1}\tan\theta_1\right)\right]} + x_{\vec{s}}\tan\left(\arctan\left(e^{n-1}\tan\theta_1\right)\right)$$

The term on the far right trivially reduces to $e^{n-1}x_{\vec{s}}\tan\theta_1$, giving us:

$$y_{\vec{s}} = \frac{\vec{a}_g x_{\vec{s}}^2}{2v_i^2(e^{2n-2}\sin^2\theta_1+\cos^2\theta_1)\cos^2\left[\arctan\left(e^{n-1}\tan\theta_1\right)\right]} + e^{n-1}x_{\vec{s}}\tan\theta_1$$

Now we work on simplifying the expression $\cos^2\left[\arctan\left(e^{n-1}\tan\theta_1\right)\right]$ in the denominator. For the time being, let $z = e^{n-1}\tan\theta_1$. It then follows that we are trying to evaluate $\cos^2(\arctan z)$ In other words, for some angle $\alpha$ such that $\tan\alpha = z$, we must find $\cos^2\alpha$. To do so, we consider the following Pythagorean identity:

$$\tan^2\alpha + 1 = \sec^2\alpha$$

Taking the reciprocal of both sides (and swapping them):

$$\cos^2\alpha = \frac{1}{\tan^2\alpha + 1}$$

Now we make the appropriate substitutions:

$$\cos^2\alpha = \frac{1}{z^2+1} \Rightarrow \cos^2\left[\arctan\left(e^{n-1}\tan\theta_1\right)\right] = \frac{1}{e^{2n-2}\tan^2\theta_1 + 1}$$

Therefore, our generalized parabolic trajectory equation is:

$$y_{\vec{s}} = \frac{\vec{a}_g x_{\vec{s}}^2}{2v_i^2(e^{2n-2}\sin^2\theta_1 + \cos^2\theta_1)\left(\frac{1}{e^{2n-2}\tan^2\theta_1+1}\right)} + e^{n-1}x_{\vec{s}}\tan\theta_1$$

Algebraically simplifying gives us:

$$y_{\vec{s}} = \frac{\vec{a}_g x_{\vec{s}}^2(e^{2n-2}\tan^2\theta_1 + 1)}{2v_i^2(e^{2n-2}\sin^2\theta_1 + \cos^2\theta_1)} + e^{n-1}x_{\vec{s}}\tan\theta_1$$

If you refer back to the Bounce Equations in the beginning of this paper, you will see that the equation we just derived above is highly similar to the actual Bounce Equation. What we're missing is a horizontal shift $S$. This shift will change for each bounce (because each bounce will be farther and farther away from the starting point). We also observed above that for the initial launch, the starting point of the launch is the starting point of the projectile itself so $S = 0$ when $n = 1$. But how do we figure out the shift for the next bounces?

The starting point of the $n$th bounce is found by summing the displacements of the bounces that came before it. For instance, if the first launch horizontally displaced a projectile 3 meters away from the origin, then the second bounce *starts* 3 meters away from the origin. If the first launch horizontally displaced a projectile 3 meters, and the second bounce horizontally displaced a projectile 3.5 meters, then the third bounce starts $3 + 3.5 = 6.5$ meters away from the origin. Recall that in a previous paper the horizontal displacement was found to be $-\frac{v_i^2\sin 2\theta}{\vec{a}_g}$ Generalizing it for every bounce and using what was discussed above tells us that for $n \geq 2$, we have:

$$S = -\sum_{k=1}^{n-1}\frac{v_{ki}^2\sin 2\theta_k}{\vec{a}_g}$$

This looks like another job for our substitutions! Letting $\theta_k = \arctan\left(e^{k-1}\tan\theta_1\right)$ and $v_{ki} = v_i\sqrt{e^{2k-2}\sin^2\theta_1 + \cos^2\theta_1}$, we have:

$$S = -\sum_{k=1}^{n-1}\frac{v_i^2(e^{2k-2}\sin^2\theta_1 + \cos^2\theta_1)\sin\left[2\arctan\left(e^{k-1}\tan\theta_1\right)\right]}{\vec{a}_g}$$

Now we take a look at the $\sin\left[2\arctan\left(e^{k-1}\tan\theta_1\right)\right]$ in the numerator. Expanding it with the double-angle identity yields:

$$\sin\left[2\arctan\left(e^{k-1}\tan\theta_1\right)\right] = 2\sin\left[\arctan\left(e^{k-1}\tan\theta_1\right)\right]\cos\left[\arctan\left(e^{k-1}\tan\theta_1\right)\right]$$

We have already found what $\cos^2\left[\arctan\left(e^{k-1}\tan\theta_1\right)\right]$ reduces to. Once again, substituting $z = e^{k-1}\tan\theta_1$, we have:

$$\cos^2(\arctan z) = \frac{1}{z^2+1} \Rightarrow \cos(\arctan z) = \frac{1}{\sqrt{z^2+1}}$$

To find $\sin(\arctan z)$, we use the rearranged Pythagorean identity $\sin\alpha = \sqrt{1-\cos^2\alpha}$:

$$\sin(\arctan z) = \sqrt{1 - \frac{1}{z^2+1}} = \sqrt{\frac{z^2}{z^2+1}} = \frac{z}{\sqrt{z^2+1}}$$

Therefore, we have:

$$\sin(2\arctan z) = 2\sin(\arctan z)\cos(\arctan z) = \frac{2z}{z^2+1} = \frac{2e^{k-1}\tan\theta_1}{e^{2k-2}\tan^2\theta_1+1}$$

Putting all of this back into our equation for $S$:

$$S = -\sum_{k=1}^{n-1} \frac{2v_i^2 e^{k-1}\tan\theta_1(e^{2k-2}\sin^2\theta_1+\cos^2\theta_1)}{\vec{a}_g(e^{2k-2}\tan^2\theta_1+1)}$$

Now notice that this expression for $S$ is positive (since $\vec{a}_g$ is negative, the sum is negative, but the negative of the sum is positive). Therefore, to produce the desired shift to the right, we must subtract $S$ from $x_{\vec{s}}$. However, we can equivalently drop the negative sign in front of the sum in the expression for $S$ and then add it to $x_{\vec{s}}$.

So finally we conclude for $n \geq 2$ we have:

$$S = \sum_{k=1}^{n-1} \frac{2v_i^2 e^{k-1}\tan\theta_1(e^{2k-2}\sin^2\theta_1+\cos^2\theta_1)}{\vec{a}_g(e^{2k-2}\tan^2\theta_1+1)}$$

For $n = 1$ we have:

$$S = 0$$

And the trajectory of the $n$th bounce is given by:

$$y_{\vec{s}} = \frac{\vec{a}_g(x_{\vec{s}}+S)^2(e^{2n-2}\tan^2\theta_1+1)}{2v_i^2(e^{2n-2}\sin^2\theta_1+\cos^2\theta_1)} + e^{n-1}(x_{\vec{s}}+S)\tan\theta_1$$

Which we can sum up simply as:

$$\boxed{y_{\vec{s}} = \frac{\vec{a}_g(x_{\vec{s}}+S)^2(e^{2n-2}\tan^2\theta_1+1)}{2v_i^2(e^{2n-2}\sin^2\theta_1+\cos^2\theta_1)} + e^{n-1}(x_{\vec{s}}+S)\tan\theta_1,\ S = \begin{cases} 0 \text{ For } n=1 \\ \sum_{k=1}^{n-1}\left(\frac{2v_i^2 e^{k-1}\tan\theta_1(e^{2k-2}\sin^2\theta_1+\cos^2\theta_1)}{\vec{a}_g(e^{2k-2}\tan^2\theta_1+1)}\right) \text{ For } n \geq 2 \end{cases}}$$

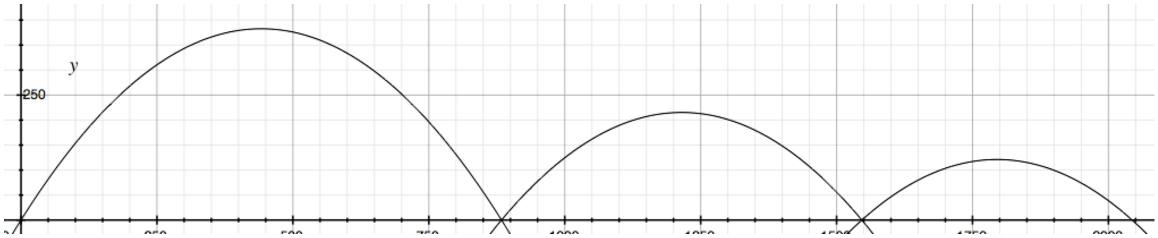


Fig 4.1: 3 bounces that were generated with the Bounce Equations. The initial velocity depicted is 100 m/s at an angle of 60˚ under Earth's gravity of −9.8 m/s$^2$ and a coefficient of restitution of 0.75.

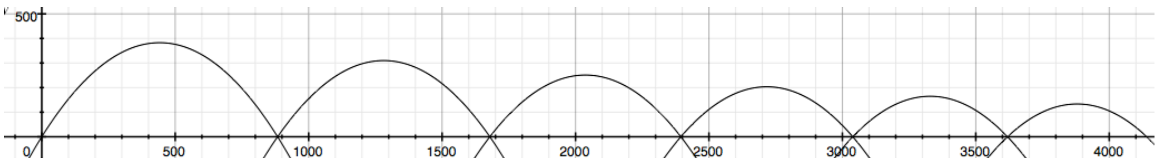


Fig 4.2: 6 bounces of the superball described in *The Superball Problem* generated with the Bounce Equations.