Problem 1: Find all pairs (x, y) of real numbers such that

$$(x+y)(x+2y) = (-x+3y)(4x-y) = 2018$$

Solution: Observe that we have:

$$(x+y)(x+2y) = (-x+3y)(4x-y)$$

Expanding:

$$x^2 + 3xy + 2y^2 = -4x^2 + 13xy - 3y^2$$

Which rearranges to:

$$5x^2 - 10xy + 5y^2 = 0$$

Which factors as:

$$5(x-y)^2 = 0$$

Hence we have $x = y$. Now our solution is clear. Upon substituting $x = y$ into our other two equations, we find:

$$(x+y)(x+2y) = 2018 \Rightarrow (2x)(3x) = 2018 \Rightarrow x = y = \pm\sqrt{\frac{1009}{3}}$$

And:

$$(-x+3y)(4x-y) = 2018 \Rightarrow (2x)(3x) = 2018$$

Which yields the same solutions. Hence our solutions (x, y) are $\boxed{\left(\frac{\sqrt{3027}}{3}, \frac{\sqrt{3027}}{3}\right)}$ and $\boxed{\left(-\frac{\sqrt{3027}}{3}, -\frac{\sqrt{3027}}{3}\right)}$.

Problem 2: Solve in real numbers the equation

$$\sqrt{x}+\sqrt{2018-x}=56$$

Solution: We square both sides first and then solve as follows:

$$\left(\sqrt{x}+\sqrt{2018-x}\right)^2=56^2$$

$$x+2\sqrt{2018x-x^2}+2018-x=3136$$
$$\sqrt{2018x-x^2}=559$$
$$x^2-2018x+559^2=0$$

Now we observe that this factors as follows:

$$(x-169)(x-1849)=0$$

Hence $\boxed{x=169}$ and $\boxed{x=1849}$. Substituting back into our original equation reveals that neither of these solutions are extraneous.

Problem 3: Find all pairs (p, q) of twin primes such that

$$(2p + q)^3 = p^3 + 2q^3 + 2018$$

Solution: If p and q are twin primes, we either have $p = q + 2$ or $q = p + 2$. We examine these cases separately.

Case 1 ($p = q + 2$): We make this substitution into our equation, which yields:

$$(3q + 4)^3 = (q + 2)^3 + 2q^3 + 2018$$

Upon expansion and rearrangement, this becomes:

$$24q^3 + 102q^2 + 132q - 1962 = 0$$

Using synthetic division, we can find that $q - 3$ is a factor. Hence we can factor the polynomial as:

$$6(q - 3)(4q^2 + 29q + 109) = 0$$

We compute the discriminant of the quadratic factor:

$$b^2 - 4ac = 29^2 - 4 \cdot 4 \cdot 109 = -903 < 0$$

Which implies that the above factorization is irreducible over $\mathbb{R}$. Hence $q = 3$ and $p = 5$ is the only valid solution in this case.

Case 2 ($q = p + 2$): We make this substitution in to our equation, which yields:

$$(3p + 2)^3 = p^3 + 2(p + 2)^3 + 2018$$

Upon expansion and rearrangement, this becomes:

$$3p^3 + 12p^2 + 24p + 2034$$

After testing all possible rational roots suggested by the Rational Root Theorem, we deduce that there are no rational roots to this equation.

Hence, the only solution (p, q) is $\boxed{(5, 3)}$.

Problem 4: Let a, b, c be nonnegative real numbers such that $a^2 + b^2 + c^2 \leq 1$. Prove that

$$(9a + 16b + 41c)(5a + 12b + 43c) < 2018$$

Solution: We observe:

$$a^2 + b^2 + c^2 \leq 1 \Rightarrow 2009(a^2 + b^2 + c^2) \leq 2009 \Rightarrow (49 + 196 + 1764)(a^2 + b^2 + c^2) \leq 2009$$

Also note that by the Cauchy-Schwarz Inequality, we have:

$$(49 + 196 + 1764)(a^2 + b^2 + c^2) \geq \left(a\sqrt{49} + b\sqrt{196} + c\sqrt{1764}\right)^2 = (7a + 14b + 42c)^2$$

Combining these two observations yields:

$$(7a + 14b + 42c)^2 \leq 2009(a^2 + b^2 + c^2) \leq 2009$$

Since $a, b, c \geq 0$, we may apply the AM-GM Inequality. This tells us that we have:

$$\sqrt{(9a + 16b + 41c)(5a + 12b + 43c)} \leq \frac{1}{2}(9a + 16b + 41c + 5a + 12b + 43c)$$

Upon simplification and rearrangement this becomes:

$$(9a + 16b + 41c)(5a + 12b + 43c) \leq (7a + 14b + 42c)^2$$

But we have already determined the maximum possible value of the RHS, namely 2009. Hence, we conclude:

$$(9a + 16b + 41c)(5a + 12b + 43c) \leq (7a + 14b + 42c)^2 \leq 2009 < 2018$$

As desired. $\square$

Problem 5: Find all pairs (a, b) of positive integers such that $a^3 - 6b^2 = 2018$ and $b^3 - 6a^2 = 155$ hold simultaneously.

Solution: We observe that by subtracting the second equation from the first, we obtain:

$$a^3 - b^3 + 6a^2 - 6b^2 = 1863$$

On the other hand, adding the two equations yields:

$$a^3 + b^3 - 6a^2 - 6b^2 = 2173$$

Both of these are Diophantine equations which we can solve by equating factors of the RHS to the factors of the polynomial on the LHS. Observe that even though $2173 > 1863$, we have $2173 = 41 \cdot 53$ and $1863 = 3^4 \cdot 23$ hence 2173 has four factors and 1863 has ten and our computation is a lot shorter if we add both equations. Unfortunately, our second polynomial which results from summing the two equations is irreducible over $\mathbb{Z}$. This forces us to subtract the equations. We continue by factoring:

$$\begin{aligned} a^3 - b^3 + 6a^2 - 6b^2 &= (a-b)(a^2 + ab + b^2) + 6(a-b)(a+b) \\ &= (a-b)(a^2 + ab + b^2 + 6a + 6b) \\ &= 1863 \end{aligned}$$

The factors of 1863 are:

$$\{1, 3, 9, 23, 27, 69, 81, 207, 621, 1863\}$$

We examine each of the ten cases separately. Let Δ denote the discriminants of the quadratic factors in the following computations. We will repeatedly apply the observation that if $f(x)$ is a quadratic function in $\mathbb{Z}[x]$, and if the roots of f are integers, then the discriminant Δ must be a perfect square.

Case 1 $(a - b = 1)$: We substitute $a = b + 1$ into the second factor to obtain:

$$\begin{aligned} a^2 + ab + b^2 + 6a + 6b &= (b+1)^2 + b(b+1) + b^2 + 6(b+1) + 6b \\ &= 3b^2 + 15b + 7 \\ &= 1863 \end{aligned}$$

Hence:

$$3b^2 + 15b - 1856 = 0$$

But we have:

$$\Delta = 15^2 + 4 \cdot 3 \cdot 1856 = 22497$$

But $\sqrt{22497} \notin \mathbb{Z}$ hence $b \notin \mathbb{Z}$ and this case does not yield a lattice point (a, b).

Case 2 $(a - b = 3)$: We substitute $a = b + 3$ into the second factor to obtain:

$$\begin{aligned} a^2 + ab + b^2 + 6a + 6b &= (b+3)^2 + b(b+3) + b^2 + 6(b+3) + 6b \\ &= 3b^2 + 21b + 27 \\ &= 621 \end{aligned}$$

Hence:

$$3b^2 + 21b - 594 = 0$$

But we have:

$$\Delta = 21^2 + 4 \cdot 3 \cdot 594 = 7569$$

We have $\sqrt{7569} = 87$, which implies that we at least have $b \in \mathbb{Q}$. Going further, we see that the quadratic actually factors as:

$$3(b + 18)(b - 11) = 0$$

Since we are are looking for solutions in $\mathbb{Z}^+$, the solution $b = -18$ is extraneous which leaves us with the solution $b = 11$ and $a = 14$ in this case.

Case 3 ($a - b = 9$)**:** We substitute $a = b + 9$ into the second factor to obtain:

$$\begin{aligned} a^2 + ab + b^2 + 6a + 6b &= (b+9)^2 + b(b+9) + b^2 + 6(b+9) + 6b \\ &= 3b^2 + 39b + 135 \\ &= 207 \end{aligned}$$

Hence:

$$3b^2 + 39b - 72 = 0$$

But we have:

$$\Delta = 39^2 + 4 \cdot 3 \cdot 72 = 2385$$

But $\sqrt{2385} \notin \mathbb{Z}$ hence $b \notin \mathbb{Z}$ and this case does not yield a lattice point (a, b).

Case 4 ($a - b = 23$)**:** We substitute $a = b + 23$ into the second factor to obtain:

$$\begin{aligned} a^2 + ab + b^2 + 6a + 6b &= (b+23)^2 + b(b+23) + b^2 + 6(b+23) + 6b \\ &= 3b^2 + 81b + 667 \\ &= 81 \end{aligned}$$

Hence:

$$3b^2 + 81b + 586 = 0$$

But we have:

$$\Delta = 81^2 - 4 \cdot 3 \cdot 586 = -471$$

But $\sqrt{-471} \notin \mathbb{Z}$ hence $b \notin \mathbb{Z}$ and this case does not yield a lattice point (a, b).

Case 5 ($a - b = 27$)**:** We substitute $a = b + 27$ into the second factor to obtain:

$$\begin{aligned} a^2 + ab + b^2 + 6a + 6b &= (b+27)^2 + b(b+27) + b^2 + 6(b+27) + 6b \\ &= 3b^2 + 83b + 891 \\ &= 69 \end{aligned}$$

Hence:

$$3b^2 + 83b + 822 = 0$$

But we have:

$$\Delta = 83^2 - 4 \cdot 3 \cdot 822 = -2975$$

But $\sqrt{-2975} \notin \mathbb{Z}$ hence $b \notin \mathbb{Z}$ and this case does not yield a lattice point (a, b).

Case 6 ($a - b = 69$)**:** We substitute $a = b + 69$ into the second factor to obtain:

$$\begin{aligned} a^2 + ab + b^2 + 6a + 6b &= (b+69)^2 + b(b+69) + b^2 + 6(b+69) + 6b \\ &= 3b^2 + 219b + 5175 \\ &= 27 \end{aligned}$$

Hence:

$$3b^2 + 219b + 5148 = 0$$

But we have:

$$\Delta = 219^2 - 4 \cdot 3 \cdot 5148 = -13815$$

But $\sqrt{-13815} \notin \mathbb{Z}$ hence $b \notin \mathbb{Z}$ and this case does not yield a lattice point (a, b).

Case 7 ($a - b = 81$)**:** We substitute $a = b + 81$ into the second factor to obtain:

$$\begin{aligned} a^2 + ab + b^2 + 6a + 6b &= (b+81)^2 + b(b+81) + b^2 + 6(b+81) + 6b \\ &= 3b^2 + 255b + 7047 \\ &= 27 \end{aligned}$$

Hence:

$$3b^2 + 255b + 7047 = 0$$

But we have:

$$\Delta = 219^2 - 4 \cdot 3 \cdot 5148 = -13815$$

But $\sqrt{-13815} \notin \mathbb{Z}$ hence $b \notin \mathbb{Z}$ and this case does not yield a lattice point (a, b).

Case 8 $(a - b = 207)$**:** We substitute $a = b + 207$ into the second factor to obtain:

$$\begin{aligned} a^2 + ab + b^2 + 6a + 6b &= (b + 207)^2 + b(b + 207) + b^2 + 6(b + 207) + 6b \\ &= 3b^2 + 633b + 44091 \\ &= 9 \end{aligned}$$

Hence:

$$3b^2 + 633b + 44082 = 0$$

But we have:

$$\Delta = 633^2 - 4 \cdot 3 \cdot 44082 = -128295$$

But $\sqrt{-128295} \notin \mathbb{Z}$ hence $b \notin \mathbb{Z}$ and this case does not yield a lattice point (a, b).

Case 9 $(a - b = 621)$**:** We substitute $a = b + 621$ into the second factor to obtain:

$$\begin{aligned} a^2 + ab + b^2 + 6a + 6b &= (b + 621)^2 + b(b + 621) + b^2 + 6(b + 621) + 6b \\ &= 3b^2 + 1875b + 389367 \\ &= 3 \end{aligned}$$

Hence:

$$3b^2 + 1875b + 389364 = 0$$

But we have:

$$\Delta = 1875^2 - 4 \cdot 3 \cdot 389364 = -1156743$$

But $\sqrt{-1156743} \notin \mathbb{Z}$ hence $b \notin \mathbb{Z}$ and this case does not yield a lattice point (a, b).

Case 10 $(a - b = 1863)$**:** We substitute $a = b + 1863$ into the second factor to obtain:

$$\begin{aligned} a^2 + ab + b^2 + 6a + 6b &= (b + 1863)^2 + b(b + 1863) + b^2 + 6(b + 1863) + 6b \\ &= 3b^2 + 5601b + 3481947 \\ &= 1 \end{aligned}$$

Hence:

$$3b^2 + 5601b + 3481946 = 0$$

But we have:

$$\Delta = 5601^2 - 4 \cdot 3 \cdot 3481946 = -10412151$$

But $\sqrt{-10412151} \notin \mathbb{Z}$ hence $b \notin \mathbb{Z}$ and this case does not yield a lattice point (a, b).

We have exhausted our set of factors and can conclude that our only solution (a, b) is $\boxed{(14, 11)}$ which we found in Case 2.

Remark: At some point we can see that our discriminants are consistently negative. Indeed, it can be shown that if we have $a - b = c$, to have $\Delta > 0$, c must satisfy the inequality:

$$-\frac{3(-7452 - 48c + c^3)}{c} > 0$$

The solution to this inequality is:

$$0 < c < r$$

Where r is the root of the function in c on the LHS of the inequality above. A computational engine provides:

$$r = \frac{1}{3}\sqrt[3]{100602 - 54\sqrt{3469745}} + \sqrt[3]{2\left(1863 + \sqrt{3469745}\right)} \approx 20.3512$$

Problem 6: Solve in positive real numbers the system of equations

$$\begin{cases} (x+3y)\sqrt{x} = 2018 - \frac{1}{3}\sqrt{x} \\ (3x+y)\sqrt{y} = 2078 + \frac{1}{3}\sqrt{x} \end{cases}$$

Solution: We let $a = \sqrt{x}$ and $b = \sqrt{y}$. Then our system becomes:

$$\begin{cases} (a^2+3b^2)a = 2018 - \frac{1}{3}a \\ (3a^2+b^2)b = 2078 + \frac{1}{3}a \end{cases}$$

Naturally, we begin by adding the two equations:

$$(a^2+3b^2)a + (3a^2+b^2)b = 4096$$

We have $4096 = 2^{12}$ which makes us suspect that the LHS can be factored as a square or maybe even a cube. Seeing the coefficients of 3 hint that it is probably a cube. Indeed, we see that this is true upon expansion:

$$a^3 + 3a^2b + 3ab^2 + b^3 = 4096$$

Which factors as:

$$(a+b)^3 = 4096$$

From this we obtain $a = 16 - b$. Substituting this into the first equation, we find obtain:

$$((16-b)^2 + 3b^2)(16-b) = 2018 - \frac{16-b}{3}$$

Expanding and rearranging:

$$-4b^3 + 96b^2 - \frac{2305}{3}b + \frac{6250}{3} = 0$$

The Rational Root Theorem yields $b = 10$ as a root. The cubic factors as:

$$-\frac{1}{3}(b-10)(12b^2 - 168b + 625)$$

The discriminant of the quadratic factor is negative, hence $b = 10$ is the only solution. Since $a + b = 16$, we must have $a = 6$ and upon reversing our substitution, we find our only solution (x, y) to be $\boxed{(36, 100)}$.

Problem 7: Let $f\colon \mathbb{Z}^+ \times \mathbb{Z}^+ \to \mathbb{Z}^+$ be a function such that $f(1,1) = 1$, $f(m+1,n) = f(m,n) + m$, and $f(m,n+1) = f(m,n) - n$ for all positive integers m and n. Find all pairs (a,b) such that $f(a,b) = 2018$.

Solution: We wish to find $a, b \in \mathbb{Z}^+$ that satisfy:

$$f(a,b) = 2018$$

Note that the only numerical value we know is $f(1,1) = 1$. So naturally, we figure that we can whittle down $f(a,b)$ to something explicitly in terms of $f(1,1)$, a, and b. Let's first reduce a down to 1 using the recursive definition provided. We have:

$$\begin{aligned}
f(a,b) &= f(a-1,b) + a - 1 \\
&= f(a-2,b) + a - 1 + a - 2 \\
&= f(a-3,b) + a - 1 + a - 2 + a - 3 \\
&\vdots \\
&= f(a-(a-1),b) + \underbrace{a + \ldots + a}_{a-1 \text{ terms}} - (1 + 2 + \ldots + (a-1)) \\
&= f(1,b) + a(a-1) - \frac{a(a-1)}{2} = 2018
\end{aligned}$$

Now we focus on reducing b in the $f(1,b)$ term:

$$\begin{aligned}
f(1,b) &= f(1,b-1) - (b-1) \\
&= f(1,b-2) - (b-1+b-2) \\
&= f(1,b-3) - (b-1+b-2+b-3) \\
&\vdots \\
&= f(1,b-(b-1)) - (\underbrace{b + \ldots + b}_{b-1 \text{ terms}}) + (1 + 2 + \ldots + (b-1)) \\
&= f(1,1) - b(b-1) + \frac{b(b-1)}{2}
\end{aligned}$$

Putting all of this together yields:

$$f(1,1) - b(b-1) + \frac{b(b-1)}{2} + a(a-1) - \frac{a(a-1)}{2} = 2018$$

Since $f(1,1) = 1$, we can eliminate it by subtracting 1 from both sides:

$$a(a-1) - \frac{a(a-1)}{2} - b(b-1) + \frac{b(b-1)}{2} = 2017$$

This simplifies as:

$$\frac{1}{2}a(a-1) - \frac{1}{2}b(b-1) = 2017$$

Now we have a two variable Diophantine polynomial! How familiar! First, we multiply by 2:

$$a(a-1) - b(b-1) = 4034$$

This expands as:

$$a^2 - a - b^2 + b = 4034$$

Now we can factor the LHS:

$$\begin{aligned}
a^2 - a - b^2 + b &= (a-b)(a+b) - (a-b) \\
&= (a-b)(a+b-1) = 4034
\end{aligned}$$

The factors of 4034 are:

$$\{\pm 1, \pm 2, \pm 2017, \pm 4034\}$$

Observe that we don't need to worry about the cases with negative factors because we can only have $a+b-1<0$ if $a+b<1$ which is not possible for $a, b \in \mathbb{Z}^+$. Thus we test the four positive cases.

Case 1 ($a-b=1$)**:** This gives us $2b=4034$ and hence $b=2017$ and $a=2018$.

Case 2 ($a-b=2$)**:** This gives us $2b+1=2017$ and hence $b=1008$ and $a=1010$

Case 3 ($a-b=2017$)**:** This gives us $2b+2016=2$ giving us a negative b which is extraneous.

Case 4 ($a-b=4034$)**:** This gives us $2b+4033=1$ giving us a negative b which is extraneous.

Hence the only solutions (a, b) are $\boxed{(1010, 1008)}$ and $\boxed{(2018, 2017)}$.

Problem 8: Solve in positive integers the equation

$$x^3 + y^3 + 3xyz = z^3 + 2018$$

Partial Solution: We substitute $w = -z$ for convenience. The equation becomes:

$$x^3 + y^3 + w^3 - 3xyw = 2018$$

Observe that this factors as:

$$(x + y + w)(x^2 + y^2 + w^2 - (xy + xw + yw)) = 2018$$

Now suppose that x, y, w are the roots of the monic cubic polynomial:

$$\mu^3 + b\mu^2 + c\mu + d = 0$$

Then by Vieta's formulas, we have:

$$-b = x + y + w$$

$$c = xy + xw + yw$$

$$-d = xyw$$

So our factorization becomes:

$$-b(b^2 - 3c) = 2018$$

Next, note that since $x, y, w \in \mathbb{Z}$, we must have $b, c, d \in \mathbb{Z}$. Hence, the above equation is Diophantine, and after some computation, we find the solutions (b, c):

$$\{(-2018, 1357441), (-2, -335), (1, 673), (1009, 339361)\}$$

Now observe that since $x, y, z > 0$, we must have $w < 0$ and $xyw < 0$ which means $d > 0$. By Descartes' Rule of Signs, for our cubic to have two positive roots, we must have exactly two sign changes between the coefficients in the cubic. Observe that with the given constraints (the cubic is monic and d is positive), this is only possible if b and c have opposing signs:

$$\mu^3 - |b|\mu^2 + |c|\mu + d$$

$$\mu^3 + |b|\mu^2 - |c|\mu + d$$

Or are both negative:

$$\mu^3 - |b|\mu^2 - |c|\mu + d$$

This means we can eliminate $(1, 673)$ and $(1009, 339361)$ for our possible pairs (b, c).

There are two ways we can go from here. We can try hunting down d. I tried doing this by equating the factorization of the original equation in terms of the coefficients of our cubic to the expansion with the sum of the cubes and the product of x, y, and w in terms of the coefficients of our cubic. I used Newton's Sums to write the sum of the cubes in terms of the coefficients of the cubic but upon writing the equation, I was getting things like $0 = 0$ which implied that I was essentially using circular reasoning. This makes sense as Newton's Sums are derived by breaking down the sum of the n^{th} powers of the roots of a polynomial into the symmetric sums of the roots.

The other approach is to proceed by consider each case remaining for (b, c) separately:

Case 1 ($b = -2018$): In this case, we have $x + y + w = 2018$. We get rid of a degree of freedom by noting $w = 2018 - (x + y)$. Now, the equation $-b(b^2 - 3c) = 2018$ becomes:

$$2018^2 - 3(xy + x(2018 - (x + y)) + y(2018 - (x + y))) = 1$$

This rearranges to:

$$3x^2 + 3xy - 6054x + 3y^2 - 6054y + 4072323 = 0$$

Now this amounts to finding lattice points on a rotated ellipse! Not sure how to do that!

Case 2 ($b = -2$): Once again, the problem boils down to finding lattice points on a rotated ellipse.

In conclusion, I am fairly certain that I'm going about this the wrong way (I highly doubt that where I'm going is the intended solution!) This seems like one of those problems that hinges on a key observation.

But aren't they all?

Problem 9: Let a and $b \in (0, \frac{\pi}{2})$ such that

$$169 \sin a \sin b + 559 \sin(a+b) + 1849 \cos a \cos b = 2018$$

Evaluate $\tan a \tan b$.

Solution: We begin by expanding the sine of the angle sum.

$$169 \sin a \sin b + 559 \sin a \cos b + 559 \sin b \cos a + 1849 \cos a \cos b = 2018$$

This equation factors as:

$$(13 \sin a + 43 \cos a)(13 \sin b + 43 \cos b) = 2018$$

Now we force some manipulations:

$$\begin{aligned}(13 \sin a + 43 \cos a)(13 \sin b + 43 \cos b) &= \frac{2018}{2018}(13 \sin a + 43 \cos a)(13 \sin b + 43 \cos b) \\ &= \frac{2018}{\left(\sqrt{2018}\right)^2}(13 \sin a + 43 \cos a)(13 \sin b + 43 \cos b) \\ &= 2018\left(\frac{13}{\sqrt{2018}} \sin a + \frac{43}{\sqrt{2018}} \cos a\right)\left(\frac{13}{\sqrt{2018}} \sin b + \frac{43}{\sqrt{2018}} \cos b\right)\end{aligned}$$

Observe that since $13^2 + 43^2 = 2018$, we have $\cos\theta = \frac{13}{\sqrt{2018}}$ and $\sin\theta = \frac{43}{\sqrt{2018}}$ where $\theta = \arctan\frac{43}{13}$. Hence, we can write:

$$\begin{aligned}2018\left(\frac{13}{\sqrt{2018}} \sin a + \frac{43}{\sqrt{2018}} \cos a\right)\left(\frac{13}{\sqrt{2018}} \sin b + \frac{43}{\sqrt{2018}} \cos b\right) &= 2018\left(\cos\theta \sin a + \sin\theta \cos a\right)\left(\cos\theta \sin b + \sin\theta \cos b\right) \\ &= 2018 \sin(a+\theta) \sin(b+\theta)\end{aligned}$$

All of this is equivalent to 2018, hence:

$$\sin(a+\theta)\sin(b+\theta) = 1$$

That is, the two sines are reciprocals of each other. However, note that the sine of any real angle is always between -1 and 1 inclusive (as this is the range of the sine function). Furthermore, since $a, b \in \left(0, \frac{\pi}{2}\right)$ and $\theta = \arctan\frac{43}{13}$, we can improve the bounds of our sines to 0 (exclusive) and 1 (inclusive). Now observe:

$$\frac{1}{n} \geq n \, \forall n \in (0, 1]$$

Which we can obtain by dividing the inequality $n \leq 1$ by n but not flipping the inequality sign (hence also implying the condition $n > 0$). Equality occurs iff $n = 1$.

We have already noted that the sine of a real angle cannot exceed 1. The only way, then, that the product of two sines can be 1 is by either both of them being 1 or -1. We can rule out the latter since $a + \theta$ and $b + \theta$ are at most in quadrant II (which still yields a positive sine). Hence the argument of the sines must be $\frac{\pi}{2}$ and we have:

$$a + \theta = b + \theta = \frac{\pi}{2}$$

This immediately yields $a = b = \frac{\pi}{2} - \arctan\frac{43}{13}$. Now all we have left is computation.

$$\tan a \tan b = \tan^2 a = \tan^2\left(\frac{\pi}{2} - \arctan\frac{43}{13}\right)$$

The tangent angle-difference formula does not work when one of the angles is $\frac{\pi}{2}$ since the tangent function is not defined for this angle. Thus we split it up into sines and cosines:

$$\tan^2\left(\frac{\pi}{2} - \arctan\frac{43}{13}\right) = \left(\frac{\sin\left(\frac{\pi}{2} - \arctan\frac{43}{13}\right)}{\cos\left(\frac{\pi}{2} - \arctan\frac{43}{13}\right)}\right)^2$$

We have:

$$\begin{aligned}\sin\left(\frac{\pi}{2} - \arctan\frac{43}{13}\right) &= \sin\frac{\pi}{2}\cos\left(\arctan\frac{43}{13}\right) - \sin\left(\arctan\frac{43}{13}\right)\cos\frac{\pi}{2} \\ &= \cos\left(\arctan\frac{43}{13}\right) \\ &= \frac{13}{\sqrt{2018}}\end{aligned}$$

And:

$$\begin{aligned}\cos\left(\frac{\pi}{2} - \arctan\frac{43}{13}\right) &= \cos\frac{\pi}{2}\cos\left(\arctan\frac{43}{13}\right) + \sin\left(\arctan\frac{43}{13}\right)\sin\frac{\pi}{2} \\ &= \sin\left(\arctan\frac{43}{13}\right) \\ &= \frac{43}{\sqrt{2018}}\end{aligned}$$

Hence our answer is:

$$\left(\frac{\frac{13}{\sqrt{2018}}}{\frac{43}{\sqrt{2018}}}\right)^2 = \left(\frac{13}{43}\right)^2 = \boxed{\frac{169}{1849}}$$

Problem 10: Prove that, for each positive integer n, 2018^n can be written as sum of three nonzero perfect squares.

Solution: Legendre's Three-Square Theorem states that a positive integer m can be written as the sum of three squares iff:

$$m \neq 4^p(8q + 7)$$

Where $p, q \in \mathbb{Z}$. Hence, we must show that 2018^n is never of this form for positive integers n. We will split this into two cases, based on the parity of n.

Case 1 (n is odd): 2018 has a prime factorization of $2 \cdot 1009$. When raised to an odd power, we see that $2 \cdot 1009$ has an odd number of factors of 2 and is therefore not a multiple of a power of 4. Hence by Legendre's Theorem, we can write 2018^n as the sum of three squares.

Case 2 (n is even): In this case, 2018^n is indeed a multiple of a power of 4 as there are an even number of factors of 2. Hence, we must show that the remaining factors of 1009, together with a product of 1009^n, cannot be expressed in the form $8q + 7$ for some integer q. This is equivalent to showing that:

$$1009^n \not\equiv 7 \pmod 8$$

But this is trivial as we observe that $1009 \equiv 1 \pmod 8$, hence:

$$1009^n \equiv 1 \not\equiv 7 \pmod 8$$

Which is enough to imply that 2018^n can be written as a sum of three squares in this case.

The issue here is that Legendre's Three-Square Theorem allows some of the squares to be 0 whereas the problem asks us to prove that 2018^n can be written as the sum of three *nonzero* squares. To do this, we proceed by induction. Our base case is:

$$44^2 + 9^2 + 1^2 = 2018$$

Now for our inductive step. Suppose that the equation $a^2 + b^2 + c^2 = 2018^k$ holds for $a, b, c, k \in \mathbb{Z}^+$. We split this into two cases, depending on the parity of k.

Case 1 (k is even):

$$\begin{aligned} 2018^{k+1} &= 2018(a^2 + b^2 + c^2) \\ &= (44^2 + 9^2 + 1^2)(a^2 + b^2 + c^2) \\ &= 44^2(a^2 + b^2 + c^2) + 9^2(a^2 + b^2 + c^2) + (a^2 + b^2 + c^2) \\ &= 44^2 \cdot 2018^k + 9^2 \cdot 2018^k + 2018^k \end{aligned}$$

If k is even, then we may write $k = 2j$ for some nonnegative integer j, and the result immediately follows:

$$\begin{aligned} 44^2 \cdot 2018^k + 9^2 \cdot 2018^k + 2018^k &= 44^2 \cdot 2018^{2j} + 9^2 \cdot 2018^{2j} + 2018^{2j} \\ &= (44 \cdot 2018^j)^2 + (9 \cdot 2018^j)^2 + (2018^j)^2 \end{aligned}$$

We have a sum of three squares, completing our induction.

Case 2 (k is odd): If k is odd, then we may write $k = 2j + 1$ for some nonnegative integer j. Now we have:

$$2018^{k+1} = 2018^2 \left(2018^{2j}\right)$$

Now we will rely upon the following lemma:

Lemma: *If a positive integer is expressible as the sum of three positive squares, then so is its square.*

Proof: Let a positive integer n be expressible as:

$$n = x^2 + y^2 + z^2$$

For positive integers x, y, z. Then:

$$\begin{aligned} n^2 &= (x^2+y^2+z^2)^2 \\ &= x^4+y^4+z^4+2x^2y^2+2x^2z^2+2y^2z^2 \end{aligned}$$

Now for some algebraic sleight-of-hand. We subtract double of the terms with a coefficient of two and add them as well yielding:

$$\begin{aligned} x^4+y^4+z^4+2x^2y^2+2x^2z^2+2y^2z^2 &= x^4+y^4+z^4-2x^2y^2-2x^2z^2-2y^2z^2+4x^2y^2+4x^2z^2+4y^2z^2 \\ &= x^4+y^4+z^4-2x^2y^2-2x^2z^2-2y^2z^2+(2xy)^2+(2xz)^2+(2yz)^2 \end{aligned}$$

This is an issue as we have three squares but extra terms! This motivates us to remove one of the squares and try to muster up a square from the remaining terms:

$$n^2 = x^4+y^4+z^4+2x^2y^2-2x^2z^2-2y^2z^2+(2xz)^2+(2yz)^2$$

The extra terms will factor! We can do this by inspection or by viewing the multivariate polynomial as univariate and then testing factors by the Factor Theorem. We have:

$$n^2 = (x^2+y^2-z^2)^2+(2xz)^2+(2yz)^2$$

Which is a sum of three squares, as desired.

Applying this lemma, we let $x^2+y^2+z^2 = 2018^2$, where of course $x, y, z \in \mathbb{Z} \setminus \{0\}$, and we have:

$$\begin{aligned} 2018^{k+1} &= 2018^2\left(2018^{2j}\right) \\ &= \left(x^2+y^2+z^2\right)\left(2018^{2j}\right) \\ &= \left(2018^j x\right)^2+\left(2018^j y\right)^2+\left(2018^j z\right)^2 \end{aligned}$$

We have a sum of three squares, which completes our induction. $\square$