The 5th AMO4 Solutions

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*Compiled from various competitions as well as original problems

SOLUTIONS

P1 Solution: Suppose to the contrary that there existed such a function f. Observe that f must have a constant term, as otherwise f is divisible by x and cannot always be prime. Suppose that this constant term is k. But then, $f(ck)$ is a multiple of k for $c \in \mathbb{Z}$, contradiction. $\square$

P2 Solution: We let v_a and v_b be the speeds of particles A and B respectively. We are given that $5v_b = 4v_a$. Let the distance traveled by the particles modulo $b-a$ be r_a and r_b:

$$r_a = v_a t \pmod{b-a}$$

The collisions always then occur where $r_a + r_b = b - a$.

Some time afterwards, particle A reaches b. Let this time be t_1. At t_1, we have $r_b = v_b t_1 = \frac{4}{5} v_a t_1$. This shows that we need a further $\frac{1}{4} t_1$ for B to reach a, at which point A will be a quarter of the way back to a (since t_1 is the time it takes for A to travel a distance of $b-a$).

Let t_2 be the time it takes to get the configuration where B is at a to the second collision. The distance traversed by both particles must be $\frac{3}{4}(b-a)$. We have:

$$t_2\left(\frac{4}{5}v_a + v_a\right) = \frac{3}{4}(b-a)$$

But notice that $\frac{4}{5}v_a t_2$, the distance traveled by B, must be $47 - a$. Therefore, we have $t_2\left(\frac{4}{5}v_a + v_a\right) = \frac{9}{5}v_a t_2 = \frac{9}{4}(47-a)$. Hence:

$$\frac{9}{4}(47-a) = \frac{3}{4}(b-a)$$

Which is:

$$2a + b = 141$$

Now we continue. A clearly reaches a before B reaches b. Note that it takes $\frac{5}{4v_a}(b-47)$ units of time for B to reach b. In that same amount of time, A must travel a distance of $\frac{5}{4}(b-47)$. After reaching a, it turns back around for a distance of:

$$\frac{5}{4}(b-47) - 47 + a = \frac{5}{4}b + a - \frac{423}{4}$$

Which puts A at a position of $\frac{5}{4}b + 2a - \frac{423}{4}$. Hence, for the third collision, the distance traveled by both particles together must be:

$$t_3\left(\frac{4}{5}v_a + v_a\right) = \frac{423}{4} - \frac{b}{4} - 2a$$

Where t_3 is the time from the point where B arrives at b to the third collision. Here, $\frac{4}{5}v_a t_3$, the distance traveled by B, must be $b - 255$. Therefore, we have $t_3\left(\frac{4}{5}v_a + v_a\right) = \frac{9}{5}v_a t_3 = \frac{9}{4}(b-255)$. Hence:

$$\frac{9}{4}(b-255) = \frac{423}{4} - \frac{b}{4} - 2a$$

Which rearranges to:

$$4a + 5b = 1359$$

Now we have the system:

$$\begin{cases} 2a + b = 141 \\ 4a + 5b = 1359 \end{cases}$$

Adding the two equations and dividing by 6 yields $a + b = \boxed{250}$.

P3 Solution: First we observe that if n has odd factors, say, $n = ox$ where o is odd, then we can write:

$$n^n + 1 = n^{ox} + 1 = (n^x)^o + 1^o$$

Which can be factored as a sum of odd powers, and hence cannot be prime. This implies that n must be a power of 2. Furthermore, the base 2 logarithm of n cannot have any odd factors because:

$$(2^{ox})^n + 1 = (2^{nx})^o + 1$$

Which can again be factored as a sum of odd powers. The powers of 2 less than or equal to 15 with no odd factors in their base 2 logarithms are 1, 2, and 4 yielding the primes 2, 5, and 257, giving us an answer of $\boxed{7}$.

P4 Solution: Since $ABCD$ is an isosceles trapezoid, both diagonals have equal length. Let $AD = BC = x$. By Ptolemy's Theorem:

$$x^2 + 35 \cdot 75 = 73^2 \Rightarrow x = 52$$

Now we observe that $\triangle PAB \sim \triangle PDC$. Letting $PA = PB = y$, we have:

$$\frac{y}{35} = \frac{y + 52}{75} \Rightarrow y = \frac{91}{2}$$

Note that the power of the point P with respect to the circumcircle of $ABCD$ (which we will express as Ω) can be expressed in two ways:

$$\operatorname{Pow}_\Omega P = PA \cdot PD = OP^2 - r^2$$

Where r is the radius of Ω. Hence, it suffices to compute r. The area of a triangle can be expressed as:

$$[ABC] = \frac{abc}{4R}$$

Where R is the circumradius of the triangle. In this case, with $\triangle ABC$, we have $R = r$, hence:

$$[ABC] = \frac{abc}{4r}$$

Since we know the side lengths of the triangle, we can compute $[ABC]$ using Heron's formula. This yields:

$$[ABC] = \sqrt{80 \cdot 45 \cdot 28 \cdot 7} = 840$$

Hence:

$$r = \frac{52 \cdot 73 \cdot 35}{4 \cdot 840} = \frac{949}{24}$$

So we conclude:

$$OP = \sqrt{r^2 + PA \cdot PD} = \sqrt{\frac{949^2}{24^2} + \frac{91}{2}\left(\frac{91}{2} + 52\right)} = \boxed{\frac{1859}{24}}$$

P5 Solution: We rely upon the following lemma:

Lemma: Let $(n+1)q > np$ be the closest multiple of $n+1$ to np. Then $p \geq q$.

Proof: Suppose that to the contrary, $(n+1)r > np$ was the closest multiple of $n+1$ to np, where $r > p$. Then, the difference between the two numbers is:

$$nr + r - np = n(r-p) + r$$

Observe that $r - p \geq 1 \Rightarrow n(r-p) \geq n$ and $r > 1$ (by implicit assumption). Adding these two inequalities yields:

$$n(r-p) + r > n + 1$$

The maximum possible distance between a multiple of n and the closest larger multiple of $n+1$ is $n+1$, and this occurs when the aforementioned multiple of n is also a multiple of $n+1$. However, we have shown that the distance between $(n+1)r$ and np is greater than $n+1$, contradicting our initial assumption that $(n+1)r$ was the closest multiple of $n+1$ that was larger than np. Hence $\exists q \leq p$ such that $(n+1)q$ is the closest larger multiple of $n+1$ to np. ■

We define a new sequence such that $p_n = \frac{a_n}{n}$ for $n > 0$. It suffices to show that $\lim_{n\to\infty} p_n$ exists.

Our lemma implies that the sequence of p_n is a decreasing sequence. We observe that if we constructed a sequence α_{n+1} by always rounding α_n *down* to the nearest n, then we must have $b_n = \frac{\alpha}{n} \leq p_n$. Under this definition of b_n, we have the recursive definition:

$$b_{n+1} = \left\lfloor \frac{n}{n+1} b_n \right\rfloor$$

We can express the floor in terms of the fractional part:

$$b_{n+1} = \frac{n}{n+1} b_n - \left\{ \frac{n}{n+1} b_n \right\}$$

Rearranging:

$$\frac{n}{n+1} b_n - b_{n+1} = \left\{ \frac{n}{n+1} b_n \right\}$$

The fractional part is between 0 (inclusive) and 1 (exclusive). Hence:

$$0 \leq \frac{n}{n+1} b_n - b_{n+1} < 1$$

The first inequality in the chain yields:

$$b_{n+1} \leq \frac{n}{n+1} b_n$$

So the sequence of b_n is decreasing. Furthermore, since $b_1 > 0$, all b_n must be positive as well, so the sequence of b_n is bounded. Hence, $\lim_{n\to\infty} b_n$ exists.

Since p_n is bounded below by b_n and is also decreasing, $\lim_{n\to\infty} p_n$ must exist as well, and we are done. □