The 5^{th} AMO4

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*Some problems were adopted from other sources.

This test consists of 5 problems, arranged from easiest to most difficult. Let n be the problem number. Then, n is also the maximum number of points possible on that problem. Hence, the maximum possible score on this test is $\sum_{k=1}^{5} k = 15$. Partial, clumsy, or non-elegant solutions on the $(n > 1)^{\text{th}}$ problem will earn a positive integer number of points less than n, the exact quantity of which will be decided by the grader. A PCNE solution on the first problem will result in 0 points.

No calculator of any kind is allowed *unless specified within the problem itself*. Compass and straightedge are allowed. $\frac{15-3n}{2}$ points will be deducted if the solver resorts to using any resources that is not his or herself on the n^{th} problem *after points have been earned if that particular solution was correct*. There is no time limit on this test. Solutions to each problem must be formal, rigorous, and LaTeXed to be scored officially.

Good luck and have fun!

PROBLEMS

Problem 1: Prove that there exists no non-constant polynomial in $\mathbb{Z}[x]$ that is prime for every positive integer x.

Problem 2: Particles A and B are at numbers a and b ($a < b$) on the real number line, respectively, and start moving towards each other at the same time. The ratio of the speed of particle A to the speed of particle B is $5 : 4$. Once either particle hits a or b, it instantaneously starts moving in the opposite direction. The two particles pass each other a second time at the number 47, and for a third time at the number 255. Find $a + b$.

Problem 3: Find the sum of all positive integers $n \leq 15$ for which $n^n + 1$ is prime.

Problem 4: Let $ABCD$ be an isosceles trapezoid with parallel bases $AB = 35$ and $CD = 75$. The length of diagonal AC is 73. Let O be the center of the circumcircle of $ABCD$ and let P be the intersection between the extensions of $\overline{BC}$ and $\overline{AD}$. The length OP can then be written as $\frac{m}{n}$ for coprime integers m and n. Find $m + n$. *A five-function calculator is permitted for use on this problem.*

Problem 5: Consider the sequences where a_{n+1} is obtained by rounding a_n to the nearest multiple of n and a_0 is a positive integer. Prove that all such sequences eventually converge to an arithmetic sequence.