The 4$^{\text{rd}}$ AMO4 Solutions

Andrew Paul *

March 14, 2018

*Compiled from various competitions as well as original problems

SOLUTIONS

P1 Solution: A decimal integer x can be written in the form:

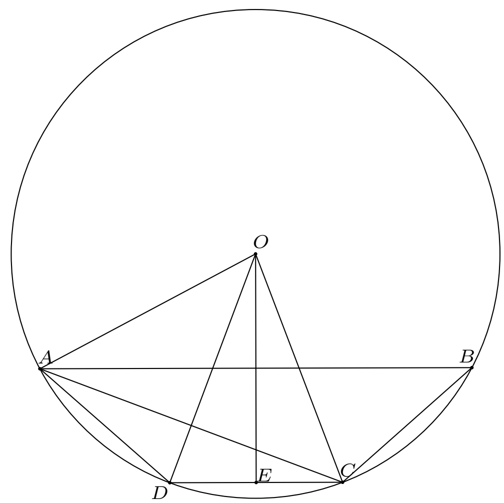
$$x = \sum_{k=0}^{n-1} a_k \cdot 10^k$$

Where n is the number of digits of x and a_k are the digits. But since $10 \equiv 1 \pmod 3$, we have:

$$x \equiv \sum_{k=0}^{n-1} a_k \pmod 3$$

As desired.

P2 Solution:



Let $\angle DOE = \theta$. First note that $\triangle OAD \cong \triangle ODC$ by SSS, hence $\angle ADO = \angle ODC = \angle ODE = 90° - \theta$. This means $\angle ADC = 180° - 2\theta$. Now, by the Law of Cosines on $\triangle ADC$:

$$AC^2 = 200^2 + 200^2 - 2 \cdot 200^2 \cos(180° - 2\theta)$$

Note that $\triangle ODE$ is a right triangle with $OD = 200\sqrt{2}$ and $DE = \frac{200}{2} = 100$. Hence, $\sin\theta = \frac{\sqrt{2}}{4}$. Observe:

$$\cos(180° - 2\theta) = -\cos(-2\theta) = -\cos 2\theta = 2\sin^2\theta - 1 = 2\left(\frac{1}{8}\right) - 1 = -\frac{3}{4}$$

So the Law of Cosines reveals:

$$AC^2 = 200^2 + 200^2 + 2 \cdot 200^2 \cdot \frac{3}{4} = 140000$$

By symmetry, $AC = BD$. Now we can conclude by Ptolemy's theorem:

$$AC^2 = 200^2 + 200AB$$

$$200^2 + 200AB = 140000 \Rightarrow \boxed{AB = 500}$$

P3 Solution: First we unnest:

$$y = \sqrt{n + \frac{n^6 - n^3}{\sqrt{n + \frac{n^6-n^3}{\sqrt{n+\ldots}}}}} = \sqrt{n + \frac{n^6 - n^3}{y}}$$

Rearranging:

$$y^3 - ny + n^3 - n^6 = 0$$

Viewing this as a cubic in y, we can factor this as:

$$(y - n^2)(y^2 + n^2 y + n^4 - n) = 0$$

We see upon substitution that $y = n^2$ is valid. Hence our summation becomes:

$$\sum_{n=20}^{170} n^2 = \sum_{n=1}^{170} n^2 - \sum_{n=1}^{19} n^2$$

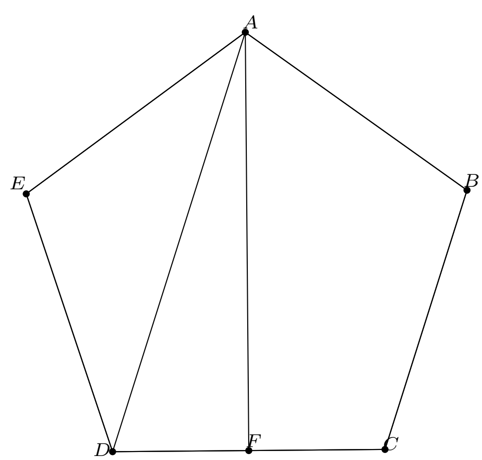
Now we apply the well-known identity:

$$\sum_{i=1}^{k} i^2 = \frac{k(k+1)(2k+1)}{6}$$

Hence the answer is:

$$\sum_{n=20}^{170} n^2 = \frac{170 \cdot 171 \cdot 341 - 19 \cdot 20 \cdot 39}{6} = \boxed{1649675}$$

P4 Solution:



Consider a regular pentagon $ABCDE$ with sides of length 1. We observe that diagonals $\overline{AD}$ and $\overline{AC}$ trisect $\angle A$. We have:

$$\angle A = \frac{180^\circ(5-2)}{5} = 108^\circ$$

So the angle is trisected into three 36° angles. To prove the trisection, we note that $\triangle EDA$ is isosceles with $\angle EDA = \angle EAD$ and:

$$\angle DEA + 2\angle EAD = 180^\circ \Rightarrow 108^\circ + 2\angle EAD = 180^\circ \Rightarrow \angle EAD = 36^\circ$$

By symmetry, we also have $\angle BAC = 36^\circ$, hence $\angle A$ is trisected as desired. Now let $x = AD$. Then by the Law of Cosines on $\triangle EDA$, we have:

$$1 = 1 + x^2 - 2x \cos 36^\circ$$

Now we drop an altitude of the pentagon from A, letting the foot be F. By symmetry, $\angle DAF = \frac{1}{2}\angle DAC = 18^\circ$. The right triangle $\triangle AFD$ gives us:

$$\sin 18^\circ = \frac{1}{2x}$$

But by the double angle formula for sine:

$$\sin 18^\circ = \sqrt{\frac{1 - \cos 36^\circ}{2}}$$

Letting $y = \cos 36^\circ$, we have the equations:

$$\begin{cases} 2xy = x^2 \\ \frac{1}{2x} = \sqrt{\frac{1-y}{2}} \end{cases}$$

Which we can treat as a two-variable system. After substitution into the second equation, we find the following cubic in y:

$$8y^3 - 8y^2 + 1 = 0$$

We can factor this as:

$$(2y - 1)(4y^2 - 2y + 1) = 0$$

Obviously we cannot have $y = \frac{1}{2}$, so we conclude with the quadratic formula, ignoring the extraneous negative root:

$$y = \cos 36^\circ = \boxed{\frac{1 + \sqrt{5}}{4}}$$

Remark: This is equivalent to $\frac{\varphi}{2}$ where φ is the golden ratio!

P5 Solution 1: We describe a general construction for even positive integers as follows.

Let n be even and let 2^{k_1} be the least power of 2 greater than n. Then:

$$n = 2^{k_1} - n_1$$

Clearly, n_1 must be even (as both n and 2^{k_1} are even). Note that since $2^{k_1 - 1} < n < 2^{k_1}$, we must have:

$$2^{k_1 - 1} - 2^{k_1} < n - 2^{k_1} < 0 \Rightarrow 2^{k_1} - 2^{k_1 - 1} > 2^{k_1} - n > 0$$

From which it follows:

$$0 < n_1 < 2^{k_1} - 2^{k_1 - 1} = 2^{k_1 - 1}$$

If n_1 is a power of 2, then we are done. Otherwise, we repeat this process on n_1.

$$n_1 = 2^{k_2} - n_2$$

Here, 2^{k_2} is the least power of 2 greater than n_1. Since we have determined $n_1 < 2^{k_1 - 1}$, we must have $2^{k_2} < 2^{k_1 - 1}$. By the monotonicity of exponential functions, this leads to $k_2 < k_1 - 1$. We may perform computations similar to those already performed to deduce $n_2 < 2^{k_2 - 1}$.

If n_2 is a power of 2, then we are done. Otherwise, we will continue this decomposition. In general, we will write:

$$n_i = 2^{k_{i+1}} - n_{i+1}$$

Where $2^{k_{i+1}}$ is the greatest power of 2 greater than n_i. We can see, as above, that $n_i < 2^{k_i - 1}$ and thus $k_{i+1} < k_i - 1$. This means that the more iterations we perform, the less even numbers we can choose for

$n_i < 2^{k_i - 1}$, until eventually, $n_i = 2$ (the minimum possible gap between powers of 2 excluding unity assuming none of the previous n_i were powers of 2). We iterate this decomposition until n_i is a power of 2, in which case we are done.

$$n = 2^{k_1} - (2^{k_2} - (2^{k_3} - (...(-2^{k_j})...))) = \sum_{i=1}^{j} (-1)^{i+1} \left(2^{k_i}\right)$$

The commutativity of addition allows us to reverse the order of the terms to satisfy the condition imposed on the absolute value of the terms (they must be in increasing order).

To construct odd numbers, we can construct $n+1$ for even numbers n. First, we must check if j is even or odd.

Case 1 (j is even) We see that since n is even, we must have $k_j \neq 0$ (else we have $2^{k_j} = 1$ and n would be odd). Furthermore, since j is even, the sign on 2^{k_j} is minus. Hence, we can simply add 1 to construct $n+1$.

Case 2 (j is odd) As before, we cannot have $2^{k_j} = 1$. However, this time we note that the sign on 2^{k_j} is plus. This means that we cannot simply add 1 (this will violate the alternating sum condition). We will thus decompose the last term:

$$2^{k_j} = 2^{k_j+1} - 2^{k_j}$$

And now our final term has a minus sign so we may add 1 to construct $n+1$. Note that no problems are encountered if $k_j + 1 = k_{j-1}$ as the alternating sign condition will then force the cancellation of $-2^{k_{j-1}}$ and 2^{k_j+1}.

Since we can construct every even integer, and every integer that exceeds any constructible even integer by one, we conclude that we can construct every positive integer in the required fashion.

P5 Solution 2 Sketch: Write the binary expansion of the number and decompose the powers of two using the fact that $2^k = 2^{k+1} - 2^k$ as needed. This decomposition process is identical to the one described in the first solution. The complete solution is left to the reader.