The 3rd AMO4 Solutions

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*Compiled from various competitions as well as original problems

SOLUTIONS

P1 Solution A1: Observe that every power of 5 has a units digit of 5 hence $5^{21}+2$ has a units digit of 7. Testing out each digit from 0 to 9, we find that no perfect square can have a units digit of 7 hence $5^{21}+2$ is $\boxed{\text{not a perfect square}}$.

P1 Solution A2: Every integer leaves a residue of either 0, 1, 2, or 3 when taken modulo 4:

$$a \equiv 0 \pmod 4$$

$$b \equiv 1 \pmod 4$$
$$c \equiv 2 \pmod 4$$
$$d \equiv 3 \pmod 4$$

Now we square these congruencies. Let n be an integer. We find that every perfect square satisfies $n \equiv 0 \pmod 4$ or $n \equiv 1 \pmod 4$. But $5^{21}+2 \equiv 3 \pmod 4$ so it is $\boxed{\text{not a perfect square}}$.

P1 Solution B1: We factor $5^{21}+1$ as a sum of cubes:

$$5^{21}+1 = \left(5^7+1\right)\left(5^{14}-5^7+1\right)$$

We have $5^7+1 = 78126$ which only has one factor of 2. The second factor is odd so it has no factors of 2. With only one factor of 2, $\boxed{5^{21}+1 \text{ cannot be a perfect square}}$.

P1 Solution B2: $5^{21}+1 \equiv 2 \pmod 4$ so it is $\boxed{\text{not a perfect square}}$.

P2 Solution: Let our 4×4 grid be a subset of a chessboard with "vertex squares" of a1, d1, d4 and a4. We will show that taking the optimal steps to avoid constructing a uniform 2×2 square still forces us to construct a 2×2 uniform square.

We begin by considering a corner, say, a1. The corners are a natural place to begin our construction as they each only have 2 adjacent squares to them. WLOG, let a1 be blue. Then a2 and b1 are also forced to be blue. Now we make b2 pink as otherwise we have constructed the uniform 2×2 square we are trying to avoid. Now a3 and c1 are forced to be blue to satisfy the coloring condition for the squares a2 and b1 respectively. But now, a4 and d1 are forced to be blue as well. Continuing in this manner, we find that the entire outer rim of the grid must be blue.

The only squares that we have not yet assigned a color to are b3, c2, and c3. Letting any one of these be blue constructs a uniform 2×2 blue square. So we let all of them be pink. But letting all of them be pink constructs a uniform 2×2 square in the center of our pulchritudinous grid. Hence, $\boxed{\text{every pulchritudinous grid must contain a uniform } 2 \times 2 \text{ square}}$.

P3 Solution: Observe that our number is of the form x^4+4y^4. This allows us to use the Sophie Germain factorization:

$$\begin{aligned} x^4+4y^4 &= x^4+4y^4+4x^2y^2-4x^2y^2 \\ &= \left(x^2+2y^2\right)^2-(2xy)^2 \\ &= \left(x^2+2xy+2y^2\right)\left(x^2-2xy+2y^2\right) \end{aligned}$$

Letting $x = 7^3$ and $y = 3$ gives us:

$$7^{12}+324 = (117649+2058+18)(117649-2058+18) = 119725 \cdot 115609$$

We can divide 119725 by 5^2 which leaves 4789. We test to see if it is divisible by any prime lesser than or equal to $\sqrt{4789}$ and it is not so 4789 is prime.

Now we see if 115609 is divisible by any prime lesser than or equal to $\sqrt{115609}$. We find that it is divisible by 13, leaving a factor of 8893. As there are no prime divisors of 8893 lesser than or equal to $\sqrt{8893}$, it must be prime. Hence:

$$7^{12} + 324 = 5^2 \cdot 13 \cdot 4789 \cdot 8893$$

Hence the sum of the factors is:

$$\left(1 + 5 + 5^2\right)(1 + 13)(1 + 4789)(1 + 8893) = 31 \cdot 14 \cdot 4790 \cdot 8894 = \boxed{18489380840}$$

P4 Solution: We complete the square:

$$\left[4\left(x - \frac{23}{2}\right)^2 + 1\right]\left[\left(y^2 - 35\right)^2 + 2017\right] = 2017$$

Now, by the Trivial Inequality, we must have:

$$4\left(x - \frac{23}{2}\right)^2 + 1 \geq 1$$

$$\left(y^2 - 35\right)^2 + 2017 \geq 2017$$

Now multiplying these inequalities yields:

$$\left[4\left(x - \frac{23}{2}\right)^2 + 1\right]\left[\left(y^2 - 35\right)^2 + 2017\right] \geq 2017$$

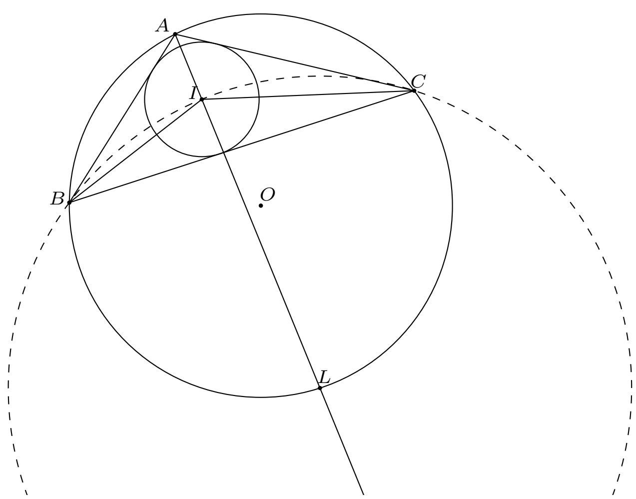
Equality occurs iff equality occurs in the first two inequalities. This happens when the squares are equal to 0. Thus:

$$x - \frac{23}{2} = 0 \Rightarrow x = \frac{23}{2}$$

$$y^2 - 35 = 0 \Rightarrow y = \sqrt{35}$$

Hence the only solution is $\boxed{(x, y) = \left(\frac{23}{2}, \sqrt{35}\right)}$.

P5 Solution:



We must show $LB = LI = LC$. We will show $LB = LI$ (the same argument can be used to show $LC = LI$).

To prove that $LB = LI$, it suffices to show $\angle IBL = \angle LIB$. Let $\angle A = 2\alpha$, $\angle B = 2\beta$, and $\angle C = 2\gamma$. Since $ABLC$ is cyclic:

$$\angle CBL = \angle CAL = \angle CAI = \alpha$$

So:

$$\angle IBL = \angle IBC + \angle CBL = \beta + \alpha$$

Finally, we note:

$$\angle LIB = 180^\circ - \angle AIB = \angle IBA + \angle BAI = \alpha + \beta$$

So $\triangle LBI$ is isosceles and $LI = LB$ as desired. *Q.E.D.*