The 2nd AMO4 Solutions

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*Compiled from various competitions as well as original problems

SOLUTIONS

P1 Solution: We desire some point on the globe such that we converge to that point by traveling a kilometer south regardless of our longitude. A little thinking gives us this:

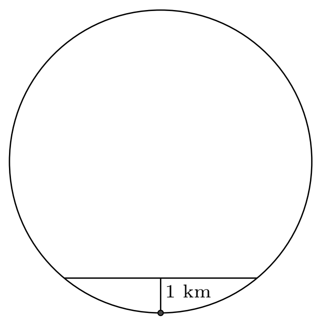


Figure 1: The South Pole is clearly the only point satisfying the conditions

On the circumference of a particular cross-section that is a little above the South Pole, our longitude is no longer relevant when we decide to move a kilometer south, because on that circumference, all souths converge to the South Pole. Since the ground is obviously snow-covered at the south pole, the answer is $\boxed{\text{white}}$.

P2 Solution:

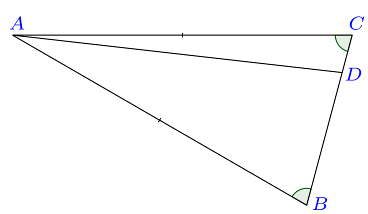


Figure 2: $\triangle ABC$ is isosceles so $\triangle ABD$ and $\triangle ACD$ satisfy ASS but are clearly not congruent

Suppose we have two triangles that satisfy ASS. All that is required is to find another pair of congruent angles or sides because both of these alone would imply that our two triangles also satisfy a valid congruency condition. With an angle and an opposite side, naturally, we consider the law of sines. Let $\angle A$, a, and b be known in $\triangle ABC$. Then:

$$\frac{\sin \angle A}{a} = \frac{\sin \angle B}{b}$$

Solving for $\angle B$:

$$\angle B = \arcsin \frac{b \sin \angle A}{a}$$

But recall that $\sin (180° - \theta) = \sin \theta$ so we may also have:

$$\angle B = 180° - \arcsin \frac{b \sin \angle A}{a}$$

This second solution is certainly a possibility for all $\triangle ABC$ satisfying $\angle A - \arcsin \frac{b \sin \angle A}{a} < 0$. *Q.E.D.*

P3 Solution 1:

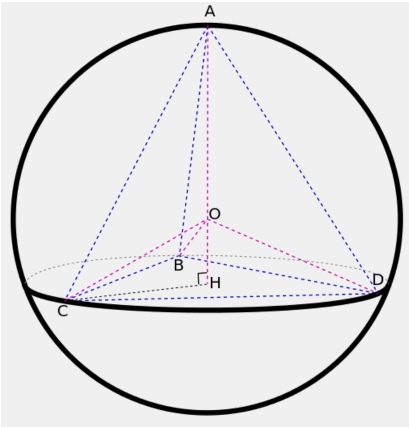


Figure 3: A regular tetrahedron

We see that $OA = OB = OC = OD = R$, the circumradius of the tetrahedron, and that $OH = r$, the inradius of the tetrahedron. Because of these equalities, we deduce:

$$\triangle OCH \cong \triangle OBH \cong \triangle ODH$$

Hence:

$$CH = BH = DH$$

It follows that H is the circumcenter of $\triangle BCD$. Furthermore, $\triangle BCD$ is equilateral so H is also the orthocenter of this triangle. Therefore, B, H, and H' (the foot of the altitude from B) are all collinear on this altitude.

Letting the side length of the tetrahedron be s, we see that $BH' = \frac{s\sqrt{3}}{2}$. Then, since H is also the centroid of $\triangle BCD$, we have:

$$HH' = \frac{1}{3} BH' = \frac{s\sqrt{3}}{6}$$

Observe that $\triangle CH'H$ is right so by the Pythagorean theorem:

$$\left(\frac{s}{2}\right)^2 + \left(\frac{s\sqrt{3}}{6}\right)^2 = CH^2 \Rightarrow CH = \frac{s\sqrt{3}}{3}$$

Now we focus on $\triangle AHC$.

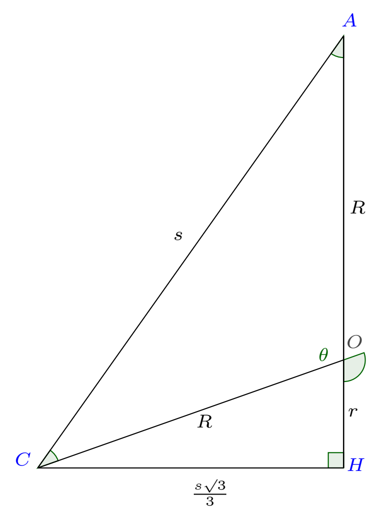


Figure 4: $\triangle AHC$

Since $\triangle OAC$ is isosceles, we have $\angle OAC \cong \angle OCA$. Furthermore, $\triangle AHC$ is right. Therefore:

$$\sin \angle OAC = \sin \angle OCA = \frac{\sqrt{3}}{3}$$

From this, we compute:

$$\sin \theta = \sin (180^\circ - 2\angle OCA) = \sin 2\angle OCA$$

We have $\cos \angle OCA = \frac{\sqrt{6}}{3}$. Therefore:

$$\sin \theta = 2\left(\frac{\sqrt{3}}{3}\right)\left(\frac{\sqrt{6}}{3}\right) = \frac{2\sqrt{2}}{3}$$

This gives us:

$$\cos \theta = -\frac{1}{3}$$

Hence $\boxed{\theta = \arccos\left(-\frac{1}{3}\right)}$.

P3 Solution 2: You may have attempted to set up the law of cosines on $\triangle AOC$:

$$s^2 = 2R^2 - 2R^2 \cos\theta$$

A little rearrangement yields:

$$\cos\theta = 1 - \frac{s^2}{2R^2}$$

So it suffices to compute $\frac{s}{R}$. To do this, we follow the steps above up to Figure 4. Then, we observe:

$$\cos\angle OCH = \cos(\angle ACH - \angle ACO) = \cos\angle ACH \cos\angle ACO + \sin\angle ACH \sin\angle ACO$$

After some Pythagorean computations, we can make the substitutions:

$$\cos\angle OCH = \left(\frac{\sqrt{3}}{3}\right)\left(\frac{\sqrt{6}}{3}\right) + \left(\frac{\sqrt{6}}{3}\right)\left(\frac{\sqrt{3}}{3}\right) = \frac{2\sqrt{2}}{3}$$

But $\triangle OCH$ is right, so we also have $\cos\angle OCH = \frac{s\sqrt{3}}{3R}$. Now:

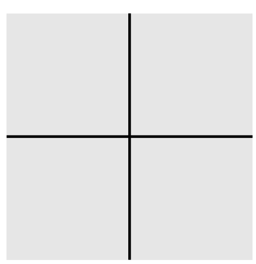
$$\frac{s\sqrt{3}}{3R} = \frac{2\sqrt{2}}{3} \Rightarrow \frac{s}{R} = \frac{2\sqrt{6}}{3}$$

Then, by our work with the law of cosines, we have:

$$\cos\theta = 1 - \frac{s^2}{2R^2} = 1 - \left(\frac{1}{2}\right)\left(\frac{24}{9}\right) = -\frac{1}{3}$$

So we conclude $\boxed{\theta = \arccos\left(-\frac{1}{3}\right)}$.

P4 Solution: We partition the square into quadrants:



To maximize the distances apart from each other between our first four points, we choose a unique quadrant for each point. But by the Pigeonhole Principle, the fifth point is then forced to share a quadrant with the point already in that quadrant. If M is the set of unattainable distances, then the distance between this fifth point and its "quadrant partner" is the desired sup M.

The largest distance in a quadrant is its diagonal, which has length $\boxed{\dfrac{\sqrt{2}}{2}}$. This is unattainable as the fifth point goes in the center of the square but its quadrant partner must be one of square's vertices.

P5 Solution: Let the sum at a certain n be S_n. Playing with the sum, we find that $S_1 = S_2 = -\frac{1}{6}$ which leads us to conjecture $S_n = -\frac{1}{6}$. Induction seems viable but daunting. Instead, we step back and continue to make some observations. We first note that the denominator can be factored:

$$2n^3 + 3n^2 + n = n(n+1)(2n+1)$$

By now, one should be roused. We have already conjectured $S_n = -\frac{1}{6}$ and now we see that the denominator of the summand is $n(n+1)(2n+1)$. This is very reminiscent of the well-known sum:

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$$

It is unclear where to continue with this, but we'll keep it in the back of our minds for now.

Next, we see that $n(n+1)(2n+1)$ is a constant, so we can pull it out of the sum:

$$S_n = \frac{1}{n(n+1)(2n+1)} \sum_{k=1}^{n} \left(2k^2 - 2nk - 3k + n + 1\right)$$

This yields:

$$S_n n(n+1)(2n+1) = \sum_{k=1}^{n} \left(2k^2 - 2nk - 3k + n + 1\right)$$

Since we suspect $S_n = -\frac{1}{6}$, we negate both sides of the equation to possibly force the LHS to be in the familiar sum-of-squares form:

$$-S_n n(n+1)(2n+1) = \sum_{k=1}^{n} \left(-2k^2 + 2nk + 3k - n - 1\right)$$

If we can show that the RHS is equal to $\sum_{k=1}^{n} k^2$, then we can finish off the problem. But how can we do this?

We factored the denominator of the summand, so let's see if the remaining numerator can be factored:

$$-2k^2 + (2n+3)k - (n+1)$$

This is quadratic in k. We seek factors of $2(n+1)$ that sum to $2n+3$. This is trivial.

$$-(k-n-1)(2k-1) = (n+1-k)(2k-1)$$

Now we have:

$$-S_n n(n+1)(2n+1) = \sum_{k=1}^{n} ((n+1-k)(2k-1))$$

This seems like a good time to start playing around with the sum on the RHS. WLOG, let n be sufficiently large for the following equation to hold:

$$\sum_{k=1}^{n} ((n+1-k)(2k-1)) = n + 3(n-1) + 5(n-2) + ... + 2n-1$$

It may not seem like it, but all hope is not lost! We *really* look at what the sum is saying:

$$\sum_{k=1}^{n} ((n+1-k)(2k-1)) = \underbrace{1+1+1+...+1}_{n \text{ times}} + \underbrace{3+3+3+...+3}_{n-1 \text{ times}} + ... + \underbrace{2n-1}_{1 \text{ time}}$$

A little clever rearranging:

$$\sum_{k=1}^{n} ((n+1-k)(2k-1)) = \underbrace{1+(1+3)+(1+3+5)+...+(1+3+5+...+2n-1)}_{n \text{ terms}}$$

Now hold on. What happens when we add the numbers within the first few parentheses?

$$\sum_{k=1}^{n} ((n+1-k)(2k-1)) = 1+4+9+...+(1+3+5+...2n-1)$$

We see sums of squares! What's going on here? It appears that:

$$\sum_{k=1}^{n} (2k-1) = n^2$$

Indeed this is true. It can be shown in a myriad of ways. Note that it suffices to show that the difference between the k^{th} square and the $(k-1)^{\text{th}}$ square is the k^{th} odd number $(2k-1)$. But this is as trivial as it gets:

$$k^2 - (k-1)^2 = 2k-1$$

Which is found by merely expanding the binomial square on the LHS! Therefore:

$$\sum_{k=1}^{n} ((n+1-k)(2k-1)) = 1+4+9+...+n^2 = \sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$$

Finally, we have:

$$-S_n n(n+1)(2n+1) = \frac{n(n+1)(2n+1)}{6} \Rightarrow \boxed{S_n = -\frac{1}{6}}$$