The 1st AMO4 Solutions

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[*]Compiled from various competitions as well as original problems

SOLUTIONS

P1 Solution: The score is minimized when one obtains help on every problem, yet provides incorrect solutions for each. 0 points are earned and $\frac{15-3n}{2}$ points are deducted on the n^{th} problem, so the minimum score possible is:

$$-\sum_{n=1}^{5}\frac{15-3n}{2}=-\frac{1}{2}\sum_{n=1}^{5}(15-3n)=-\frac{75}{2}+\frac{3}{2}\sum_{n=1}^{5}n=\frac{45}{2}-\frac{75}{2}=\boxed{-15}$$

P2 Solution 1: The LHS of each equation are *almost* the 3 symmetric sums of x, y, and z which immediately motivates a solution using Vieta. What's troubling us are the "$-$" signs that appear in the first and second equations. Wishful thinking leads us to the substitution $w=-z$, which yields:

$$\begin{cases} x+y+w=0 \\ wx+xy+yw=-27 \\ xyw=-54 \end{cases}$$

Aha! Now, by Vieta, x, y, and w are roots of the polynomial:

$$t^3-27t+54$$

Rational root theorem and synthetic division gives a factorization of:

$$(t+6)(t-3)^2$$

Hence $(x,y,w)\in\{(3,3,-6),(3,-6,3),(-6,3,3)\}$. Now, we reverse our substitution to find:

$$\boxed{(x,y,z)\in\{(3,3,6),(3,-6,-3),(-6,3,-3)\}}$$

P2 Solution 2: The RHS of the first equation is particularly convenient. We have $x+y=z$, and then by substitution into the second equation, we have $(x+y)x-xy+(x+y)y=27$ or:

$$(x+y)^2-xy=27$$

Furthermore, by the third equation, we have:

$$(x+y)xy=54$$

Let $\{x,y\}$ be the roots of t^2+bt+c. Then by Vieta:

$$\begin{cases} b^2-c=27 \\ -bc=54 \end{cases}$$

The system has solutions

$$(b, c) \in \{(-6, 9), (3, -18)\}$$

So our quadratic is either $t^2 - 6t + 9 = (t-3)^2$ or $t + 3t - 18 = (t+6)(t-3)$. Hence the primitive solutions are $(x, y) \in \{(3, 3), (-6, 3)\}$. The last "hidden" solution is the permutation of the last ordered pair which gives $(3, -6)$. Since $x + y = z$, we conclude:

$$\boxed{(x, y, z) \in \{(3, 3, 6), (3, -6, -3), (-6, 3, -3)\}}$$

P3 Solution: By binomial theorem:

$$(2x + 3y)^{20} = k_1 x^{20} + k_2 x^{19} y + ... + k_{20} x y^{19} + k_{21} y^{21}$$

Observe that by letting $x = y = 1$, we have:

$$(2 + 3)^{20} = 5^{20} = k_1 + k_2 + ... + k_{20} + k_{21} = \sum_{i=1}^{21} k_i = a + b + c$$

So now we must count the number of ways we can partition 5^{20} into 3 nonnegative integers. To do so, we use *sticks and stones.* Consider 2 sticks and 5^{20} stones, which I will represent as a binary number:

$$11\,\underbrace{000...0}_{5^{20}\text{ zeros}}$$

Each digit of 1 is a stick and 0 is a stone. The number of stones to the left of the leftmost stick represents the value of a, the number of stones between the two sticks represents b, and the number of stones to the right of the rightmost stick represents c. In the above configuration, we have $a = b = 0$ and $c = 5^{20}$.

The number of nonnegative integer triples (a, b, c) is then the number of ways we can arrange the digits of the above binary number. This is a simple counting problem: treat every digit as distinguishable and then divide out the number of permutations of each set of indistinguishable digits. This gives us:

$$\frac{\left(5^{20} + 2\right)!}{2! \cdot 5^{20}!} = \frac{\left(5^{20} + 1\right)\left(5^{20} + 2\right)}{2}$$

You like big integers? Brace yourself.

$$= \boxed{4,547,473,508,864,784,240,722,656,251}$$

P4 Solution: This result is well-known. The point of concurrency is known as the *radical center* of the circles. To show that all three chords are concurrent, we will first show that the three chords are subsets of their respective *radical axes.*

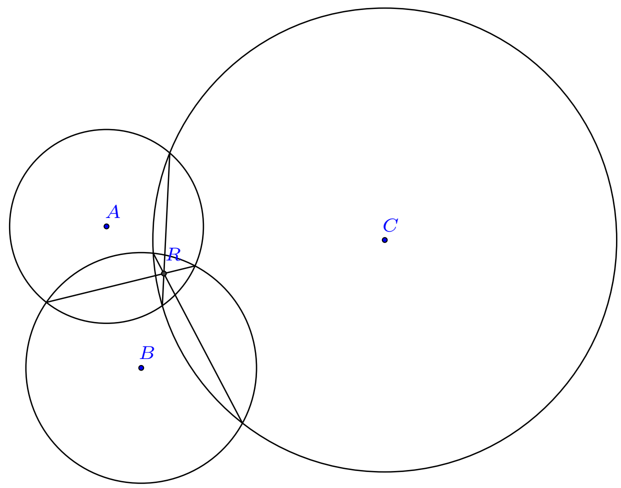

Figure 1: The common chords are concurrent at R

We define the radical axis of two circles as the locus of points that have equivalent powers with respect to both circles. That is, for circles A and B, the radical axis is all points P such that:

$$\text{Pow}_A(P) = \text{Pow}_B(P)$$

It can be shown that the radical axis (as its name suggests) is a straight line. Observe that when two circles, A and B, intersect twice, then both points of intersection, P_1 and P_2, must lie on the radical axis. This is because:

$$\text{Pow}_A(P_1) = \text{Pow}_B(P_1) = \text{Pow}_A(P_2) = \text{Pow}_B(P_2) = 0$$

As P_1 and P_2 are on the circumferences of both circles. Therefore, the common chord between circles A and B is a subset of the radical axis of those circles. This applies to all pairs of circles in our configuration, as we chose circles A and B WLOG (see Figure 1). Consider the intersection of the radical axes of A and B and B and C. Let this point be R. Then, it follows that we have:

$$\text{Pow}_A(R) = \text{Pow}_B(R) = \text{Pow}_C(R)$$

From the LHS and RHS of this equality, we deduce that R lies on the radical axis of A and C, of which the common segment between A and C is a subset of. Therefore, all three chords concur at R as desired. *Q.E.D.*

P5 Solution: We wish to find some optimal combination of rowing and walking. Note that by the symmetry of the circular lake, it does not matter if Ravi walks first and then rows, or rows first and then walks. Furthermore, alternating rowing and walking is clearly not beneficial. Therefore, we can assume WLOG that Ravi rows first to some point on the circle, and walks the rest of the way around the circle. Letting θ be the angle formed by the radius with an endpoint that is Ravi's starting point A and the radius with an endpoint that is where Ravi stops rowing B, we have an isosceles triangle whose legs are $r = \frac{1}{2}$. The side opposite to the enclosed angle θ is the length that Ravi will row:

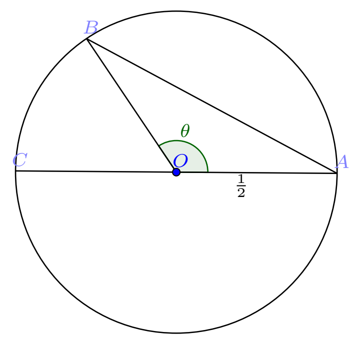

Figure 2: Ravi will row across $\overline{AB}$ and will walk along $\widehat{BC}$

By the law of cosines, we find:

$$AB = \sqrt{\frac{1}{2} - \frac{1}{2}\cos\theta}$$

Also observe that we can make the restriction $\theta \in [0, \pi]$ because the symmetry of the circle will allow us to account for any other angle with an angle in this interval. It follows that the time taken for rowing is:

$$t_r(\theta) = \frac{\sqrt{\frac{1}{2} - \frac{1}{2}\cos\theta}}{5}$$

Likewise, we see that $\widehat{BC} = \frac{\pi - \theta}{2}$ so the time it takes to walk along the circumference is:

$$t_w(\theta) = \frac{\pi - \theta}{14}$$

Hence the total time travelled is:

$$\Sigma t(\theta) = t_r(\theta) + t_w(\theta) = \frac{\sqrt{\frac{1}{2} - \frac{1}{2}\cos\theta}}{5} + \frac{\pi - \theta}{14}$$

We differentiate WRT θ:

$$\Sigma t'(\theta) = \frac{\sin\theta}{20\sqrt{\frac{1}{2} - \frac{1}{2}\cos\theta}} - \frac{1}{14}$$

Now observe that $\Sigma t'(\theta) = 0$ has an approximate solution of $\theta = \frac{\pi}{2}$ since $\Sigma t'\left(\frac{\pi}{2}\right) = \frac{7\sqrt{2}-10}{140} \approx 0$. In fact, we can find that the only solution of $\Sigma t'(\theta) = 0$ is $\arccos \frac{1}{49}$.

We wish to apply the first derivative test for the critical point $\theta = \arccos \frac{1}{49}$. We observe that since $\arccos \frac{1}{49} \approx \frac{\pi}{2}$, we must have $\frac{\pi}{6} < \arccos \frac{1}{49} < \frac{5\pi}{6}$. We can now see:

$$\Sigma t'\left(\frac{\pi}{6}\right) > 0 > \Sigma t'\left(\frac{5\pi}{6}\right)$$

Which implies that $\theta = \arccos \frac{1}{49}$ is a relative maximum in the interval $[0, \pi]$. Henceforth we can say with certainty that $\Sigma t(\theta)$ is minimized when $\theta = 0$ or $\theta = \pi$. Evaluating $\Sigma t(\theta)$ at both of these angles, we find that it is minimized at $\theta = \pi$. Note that on Figure 2, this implies $B = C$, whereas $\theta = 0$ would imply $B = A$.

In other words, Ravi minimizes his time spent trying to reach the other side of the lake by simply rowing straight across the diameter of the lake without walking at all. The diameter that must be traversed has a length of 1 km, and at a speed of 5 km/hour, he will cross the lake in $\boxed{\text{12 minutes}}$.