Here, I will be presenting my solutions to the problems which I have solved from *100 Geometry Problems: Bridging the Gap from AIME to USAMO* by David Altizio. This is a compilation that will grow over time.

1. [MAΘ ????] In the figure shown below, circle B is tangent to circle A at X, circle C is tangent to circle A at Y, and circles B and C are tangent to each other. If $AB = 6$, $AC = 5$, and $BC = 9$, what is AX?

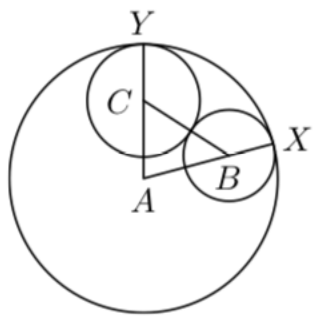

Figure 1: Problem 1

Solution: Let $BX = x$ and $CY = y$. Suppose that the point of tangency between the circles B and C be D. Then, it is known that $AX = AY$, $BD = BX$, and $CD = CY$ as these are radii of their respective circles. Then, we have:

$$\begin{cases} x + y = 9 \\ x + 6 = y + 5 \end{cases}$$

Solving this system, we find that $x = 4$ and $AB + x = 6 + 4 = \boxed{10}$.

2. [AHSME ????] In triangle ABC, $AC = CD$ and $\angle CAB - \angle ABC = 30°$. What is the measure of $\angle BAD$?

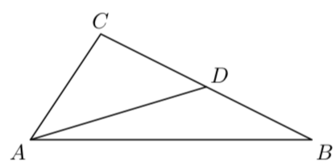

Figure 2: Problem 2

Solution: Let $\angle CAD = \theta$ and $\angle DAB = \gamma$. Since $\triangle ACD$ is isosceles, we must have $\angle ADC = \theta$ and it follows that $\angle ACD = 180° - 2\theta$ and $\angle ADB = 180° - \theta$. Setting the sum of the angles of $\triangle ABC$ equal to $180°$, we find that $\angle ABC = \theta - \gamma$. We are given that $\angle ABC = \theta + \gamma - 30°$. Setting these two equal yields:

$$\theta - \gamma = \theta + \gamma - 30° \Rightarrow \gamma = \boxed{15°}$$

3. [AMC 10A 2004] Square $ABCD$ has side length 2. A semicircle with diameter AB is constructed inside the square, and the tangent to the semicircle from C intersects side AD at E. What is the length of CE?

Solution: Let the center of the semicircle be F and the point of tangency between the semicircle and $\overline{CE}$ be G.

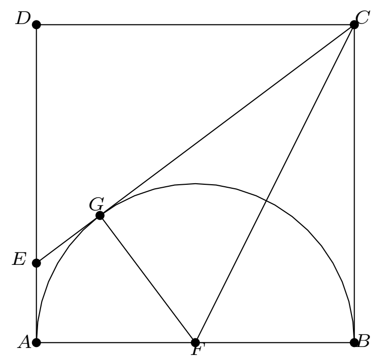

Figure 3: Problem 3

We observe that $\angle CBF = \angle CGF = 90^\circ$, and $FB = FG$, so $\triangle CGF \cong \triangle CBF$ and $CG = CB = 2$.

Next, we see that $EG^2 = EA^2$, the power of point E with respect to the semicircle, hence $EG = EA$. Let us call this length x. It follows that:

$$CE = x + 2$$

$$DE = 2 - x$$

Now we finish by the Pythagorean Theorem on $\triangle CDE$:

$$4 + (2 - x)^2 = (x + 2)^2$$

This has the solution $x = \frac{1}{2}$, hence $CE = \frac{1}{2} + 2 = \boxed{\frac{5}{2}}$.

4. [**AMC 10B 2011**] Rectangle $ABCD$ has $AB = 6$ and $BC = 3$. Point M is chosen on side AB so that $\angle AMD = \angle CMD$. What is the degree measure of $\angle AMD$?

Solution:

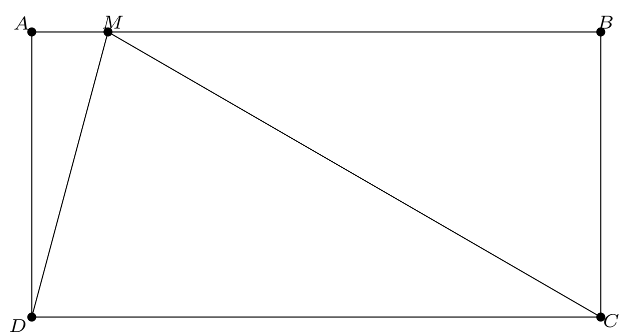

Figure 4: Problem 4

Let $\angle AMD = \angle CMD = \theta$. Then, since $\triangle AMD$ is right, we have $\angle ADM = 90^\circ - \theta$, from which it follows, $\angle CDM = \theta$. Since $\angle CDM = \angle CMD = \theta$, $\triangle CDM$ is isosceles.

This means that $CM = 6$, which makes $\triangle BCM$ a right triangle with a leg of 3 and a hypotenuse of 6, implying that it is a $30^\circ - 60^\circ - 90^\circ$ triangle with $\angle BCM = 60^\circ$. We continue the angle chase: $\angle DCM = 180^\circ - 2\theta$ and $\angle BCM = 2\theta - 90^\circ$. Now:

$$2\theta - 90^\circ = 60^\circ \Rightarrow \theta = \boxed{75^\circ}$$

5. [**AIME 2011**] On square $ABCD$, point E lies on side AD and point F lies on side BC, so that $BE = EF = FD = 30$. Find the area of the square.

Solution: Let G be the foot of the altitude from F to $\overline{AD}$.

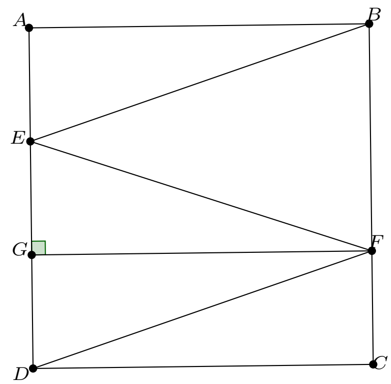

Figure 5: Problem 5

By symmetry, $\triangle ABE \cong \triangle CDF$. Since $GF = AB$ and $EF = EB$, we must also have $\triangle ABE \cong \triangle GFE$ by HL congruence. From this it follows:

$$AE + EG + GD = 3AE = AD$$

Letting x be the side length of the square, we apply the Pythagorean Theorem to $\triangle ABE$:

$$\frac{x^2}{9} + x^2 = 900$$

Which yields $x^2 = \boxed{810}$.

6. Points A, B, and C are situated in the plane such that $\angle ABC = 90°$. Let D be an arbitrary point on $\overline{AB}$, and let E be the foot of the perpendicular from D to $\overline{AC}$. Prove that $\angle DBE = \angle DCE$.

Solution: This doesn't deserve a diagram. $\angle CED = \angle CBD = 90°$ and quadrilateral $CBDE$ is cyclic. $\square$

7. [**AMC 10B 2012**] Four distinct points are arranged in a plane so that the segments connecting them have lengths a, a, a, a, $2a$, and b. What is the ratio of b to a?

Solution: Suppose that the points are A, B, C, and D. First, we consider a rhombic configuration. Suppose, WLOG, that $AB = BC = CD = AD = a$. This leaves the diagonals of the rhombus, AC and BD to be the lengths $2a$ and b. WLOG, suppose that $AC = 2a$. However, by the triangle inequality on $\triangle ABC$, we must have $AC > 2a$, contradiction.

This then forces $\triangle ABC$ into degeneracy, causing A, B, and C to be collinear. We complete the configuration by placing D such that (WLOG) $\triangle ABD$ is equilateral.

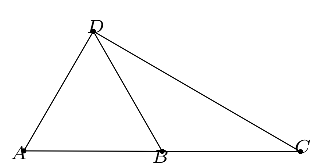

Figure 6: In this configuration, $AB = BC = AD = BD = a$, $AC = 2a$, and $CD = b$.

Now that we have our configuration, the rest is a computation. Since $\angle ABD = 60°$, we have $\angle CBD = 120°$. Now, by the Law of Cosines on $\triangle BCD$, we have:

$$b^2 = 2a^2 - 2a^2 \cos 120° \Rightarrow b^2 = 3a^2$$

From which it follows, $\frac{b}{a} = \boxed{\sqrt{3}}$.

8. [Britain 2010] Let ABC be a triangle with $\angle CAB$ a right angle. The point L lies on the side BC between B and C. The circle BAL meets the line AC again at M and the circle CAL meets the line $\overleftrightarrow{AB}$ again at N. Prove that L, M, and N lie on a straight line.

Solution: We employ phantom points. Let the extension of $\overline{ML}$ intersect the extension of $\overline{AB}$ at N'. It suffices to show that $N' = N$.

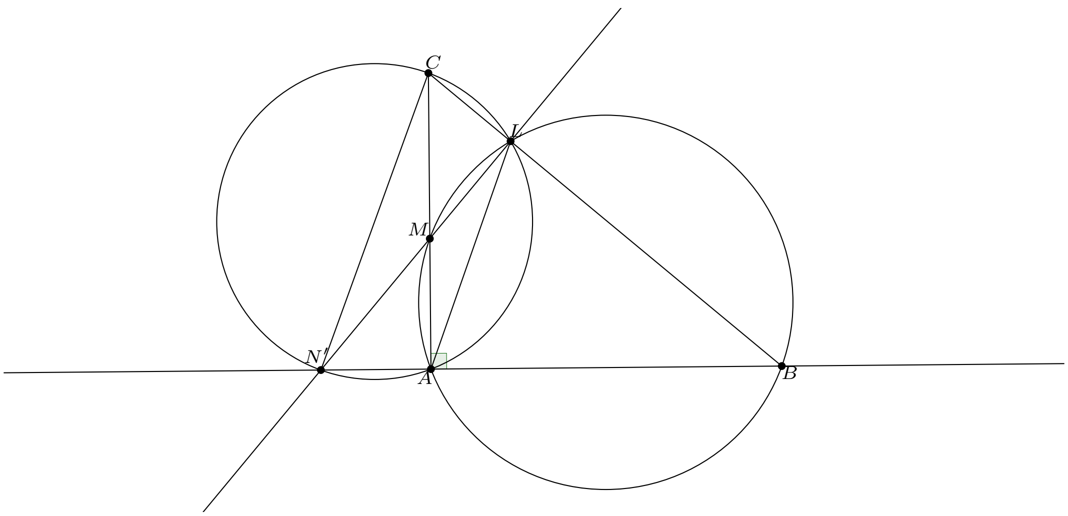

Figure 7: It suffices to show that $N' = N$

Let $\angle ACB = \theta$. Then, $\angle ABC = 90° - \theta$. Since quadrilateral $ABLM$ is cyclic, $\angle AML = 90° + \theta$. This makes $\angle AMN = 90° - \theta$ and since $\triangle AMN$ is a right triangle, $\angle ANL = \theta$.

Since $\angle ANL = \angle ACL = \theta$, quadrilateral $ALCN$ is cyclic, and $N' = N$ as desired. $\square$

9. [OMO 2014] Let ABC be a triangle with incenter I and $AB = 1400$, $AC = 1800$, $BC = 2014$. The circle centered at I passing through A intersects line BC at points X and Y. Compute the length XY.

Solution: We draw the incircle of $\triangle ABC$. Let D be the point of tangency between the incircle and $\overline{BC}$ and let F be similarly defined but for side $\overline{AB}$.

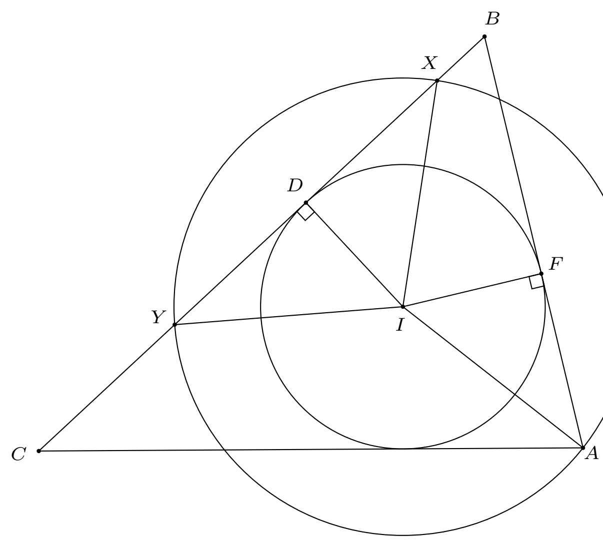

Figure 8: Problem 9

We use a lemma to compute AF. In particular, this lemma states that:

$$AF = s - BC$$

Where $s = 2607$ is the semiperimeter of $\triangle ABC$. This lemma is easily proven by setting up a system of equations with the portions of the sides of $\triangle ABC$ formed by the contact triangle. Hence, $AF = 593$. By Heron's formula:

$$[ABC] = \sqrt{2607 \cdot 593 \cdot 1207 \cdot 807} = 3\sqrt{167314669511}$$

Let $r = IF$, the inradius. Then, this is also equal to rs. Hence, $r = \frac{\sqrt{167314669511}}{869}$. Then by the Pythagorean theorem on $\triangle AFI$,

$$AI = \sqrt{593^2 + \frac{167314669511}{869^2}} = \frac{200\sqrt{10821657}}{869}$$

This is a radius of the larger circle along with $\overline{XI}$ and $\overline{YI}$. By the Pythagorean theorem on $\triangle XDI$:

$$XD = \sqrt{\left(\frac{200\sqrt{10821657}}{869}\right)^2 - \frac{167314669511}{869^2}} = 593$$

And finally:

$$XY = 2XD = \boxed{1186}$$

10. [**India RMO 2014**] Let ABC be an isosceles triangle with $AB = AC$ and let Γ denote its circumcircle. A point D is on arc AB of Γ not containing C. A point E is on arc AC of Γ not containing B. If $AD = CE$ prove that BE is parallel to AD.

Solution:

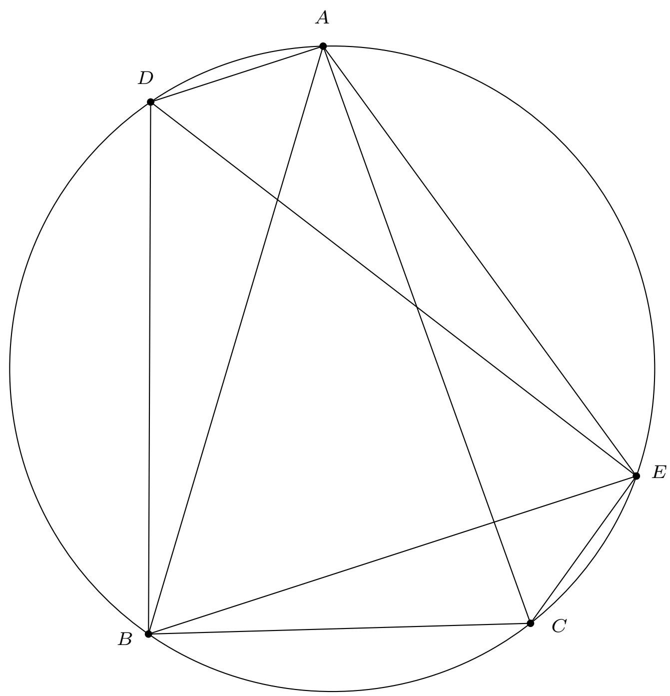

Figure 9: Problem 10

Since $\triangle ABC$ is isosceles, $\overset{\frown}{AB} = \overset{\frown}{AC}$. Furthermore, $AD = CE$, hence $\overset{\frown}{AD} = \overset{\frown}{CE}$, and $\overset{\frown}{BD} = \overset{\frown}{AB} - \overset{\frown}{AD} = \overset{\frown}{AC} - \overset{\frown}{CE} = \overset{\frown}{AE}$. This implies $\angle BAD = \angle ADE$.

But $\angle ADE = \angle ABE$ because $ADBE$ is cyclic and we are done. $\square$

11. A closed planar shape is said to be *equiable* if the numerical values of its perimeter and area are the same. For example, a square with side length 4 is equiable since its perimeter and area are both 16. Show that any closed shape in the plane can be dilated to become equiable. (A dilation is an affine transformation in which a shape is stretched or shrunk. In other words, if $\mathcal{A}$ is a dilated version of $\mathcal{B}$ then $\mathcal{A}$ is similar to $\mathcal{B}$).

Solution: Some of the basic properties of a homothety with scale factor k are that distances between two points on the figure will be scaled by k, and hence if the figure is closed, the area will be scaled by k^2. Suppose P and A are the perimeter and area respectively of a closed planar shape. Then we wish to apply a homothety with scale factor k such that:

$$k^2 A = kP$$

Therefore, there exists a class of homotheties, namely those with scale factor $k = \frac{P}{A}$, or the degenerate case of $k = 0$ (which is also a solution to the equation above), that maps every closed planar shape to a similar one that is equiable. $\square$

12. [**David Altizio**] Triangle AEF is a right triangle with $AE = 4$ and $EF = 3$. The triangle is inscribed inside square $ABCD$ as shown. What is the area of the square?

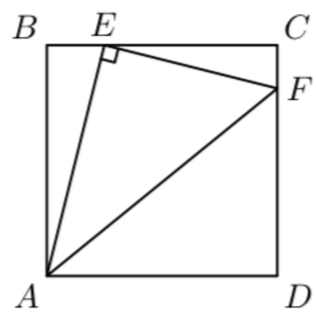

Figure 10: Problem 12

Solution: $AF = 5$. We have $\angle CEF = 180° - 90° - \angle AEB = 90° - \angle AEB$, hence $\triangle ABE \sim \triangle ECF$. Hence:

$$\frac{BE}{4} = \frac{CF}{3} \Rightarrow CF = \frac{3}{4}BE$$

Let s be the side length of the square, and let $x = BE$. Then, by the Pythagorean theorem on $\triangle ADF$:

$$s^2 + \left(s - \frac{3}{4}x\right)^2 = 25$$

By the Pythagorean theorem on $\triangle ABE$:

$$s^2 + x^2 = 16$$

Solving this system yields $s^2 = \boxed{\frac{256}{17}}$.

13. Points A and B are located on circle Γ, and point C is an arbitrary point in the interior of Γ. Extend $\overline{AC}$ and $\overline{BC}$ past C so that they hit Γ at M and N respectively. Let X denote the foot of the perpendicular from M to $\overline{BN}$, and let Y denote the foot of the perpendicular from N to $\overline{AM}$. Prove that $\overline{AB} \parallel \overline{XY}$.

Solution:

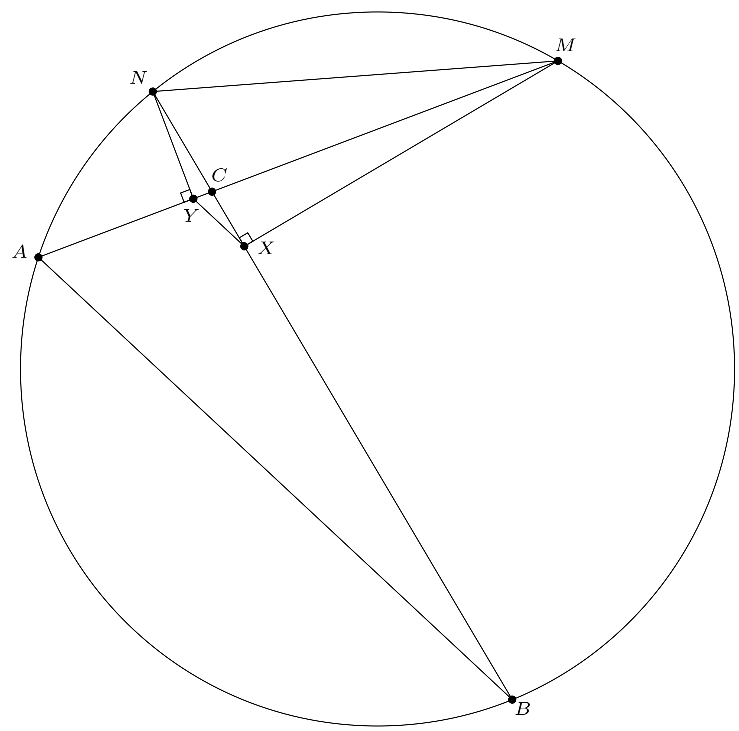

Figure 11: Problem 13

We observe that $\angle BAM = \angle BNM$ as both open up to the same arc. Let this angle be α. Next, we observe that since $\triangle MXN$ is a right triangle, we must have $\angle XMN = \angle 90° - \alpha$. Furthermore, since $\angle NYM = \angle MXN = 90°$, $MNYX$ is cyclic. Hence:

$$\angle NYX + \angle NMX = 180° \Rightarrow \angle NYX = 90° + \alpha$$

And since $\angle NYM = 90°$, we must have $\angle CYX = \alpha$. Using a similar argument, or by summing the angles in $\triangle CYX$, we can deduce $\angle CXY = \angle CBA$.

Hence, $\overline{AB} \parallel \overline{XY}$. $\square$

14. [AIME 2007] Square $ABCD$ has side length 13, and points E and F are exterior to the square such that $BE = DF = 5$ and $AE = CF = 12$. Find EF^2.

Solution: We let the foot of the altitude from E be X. Observe that $(5, 12, 13)$ is a Pythagorean triple, so $\triangle AEB$ and $\triangle CFD$ are right triangles. Then, the area of $\triangle AEB$ is given by:

$$[\triangle AEB] = \frac{1}{2} \cdot 5 \cdot 12 = \frac{1}{2} \cdot 13 \cdot EX \Rightarrow EX = \frac{60}{13}$$

From this, we use the Pythagorean theorem on $\triangle EXB$ to compute $BX = \frac{25}{13}$. Hence, the vector $\mathbf{EF}$ is given by:

$$\mathbf{EF} = \left\langle 2 \cdot \frac{25}{13}, 13 + 2 \cdot \frac{60}{13} \right\rangle = \left\langle \frac{50}{13}, \frac{289}{13} \right\rangle$$

And:

$$EF^2 = \frac{2500 + 83521}{169} = \boxed{509}$$

15. Let Γ be the circumcircle of $\triangle ABC$, and let D, E, F be the midpoints of arcs AB, BC, CA respectively. Prove that $\overline{DF} \perp \overline{AE}$.

Solution:

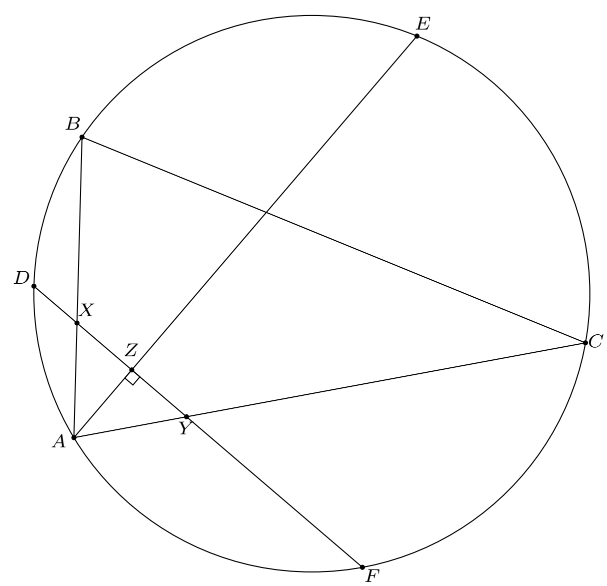

Figure 12: Problem 15

We let $X = \overline{AB} \cap \overline{DF}$, $Y = \overline{AC} \cap \overline{DF}$, and $Z = \overline{AE} \cap \overline{DF}$.

Since $\widehat{BE} = \widehat{CE}$, $\angle BAE = \angle CAE$. Furthermore, $\angle AYZ = \frac{1}{2}\left(\widehat{AD} + \widehat{CF}\right)$ and $\angle AXZ = \frac{1}{2}\left(\widehat{BD} + \widehat{AF}\right)$. But $\widehat{AD} = \widehat{BD}$ and $\widehat{AF} = \widehat{CF}$, so $\angle AXZ = \angle AYZ$.

This establishes $\triangle AZX \sim \triangle AZY$, from which it follows $\angle AZX = \angle AZY$. $\square$

16. **[AIME 1984]** In tetrahedron $ABCD$, edge AB has length 3 cm. The area of face ABC is 15 cm^2 and the area of face ABD is 12 cm^2. These two faces meet each other at a 30° angle. Find the volume of the tetrahedron in cm^3.

Solution: Cavalieri's principle implies that we may shift the apex A to any location and preserve the volume of $ABCD$ as long as the distance from A to the plane BCD is preserved. So let us shift A such that $AB = h$, the height of the tetrahedron. In this case, $\angle ABC = \angle ABD = 90°$, since $\overline{AB}$ must be perpendicular to the plane BCD. We have:

$$[\triangle ABC] = \frac{1}{2} \cdot 3 \cdot BC = 15$$

And:

$$[\triangle ABD] = \frac{1}{2} \cdot 3 \cdot BD = 12$$

From these we obtain $BC = 10$ and $BD = 8$, respectively. Now the area $\triangle BCD$ is given by:

$$[\triangle BCD] = \frac{1}{2} \cdot 15 \cdot 12 \sin 30° = 45$$

And the volume of $ABCD$ is thus $\frac{1}{3}[\triangle BCD]h = \frac{1}{3} \cdot 3 \cdot 45 = \boxed{45 \text{ cm}^3}$.

17. Let $P_1P_2P_3P_4$ be a quadrilateral inscribed in a circle with diameter of length D, and let X be the intersection of its diagonals. If $\overline{P_1P_3} \perp \overline{P_2P_4}$ prove that

$$D^2 = XP_1^2 + XP_2^2 + XP_3^2 + XP_4^2$$

Solution: Let the center of the circle be O and let the diameter through P_1 intercept the circle a second time at Y.

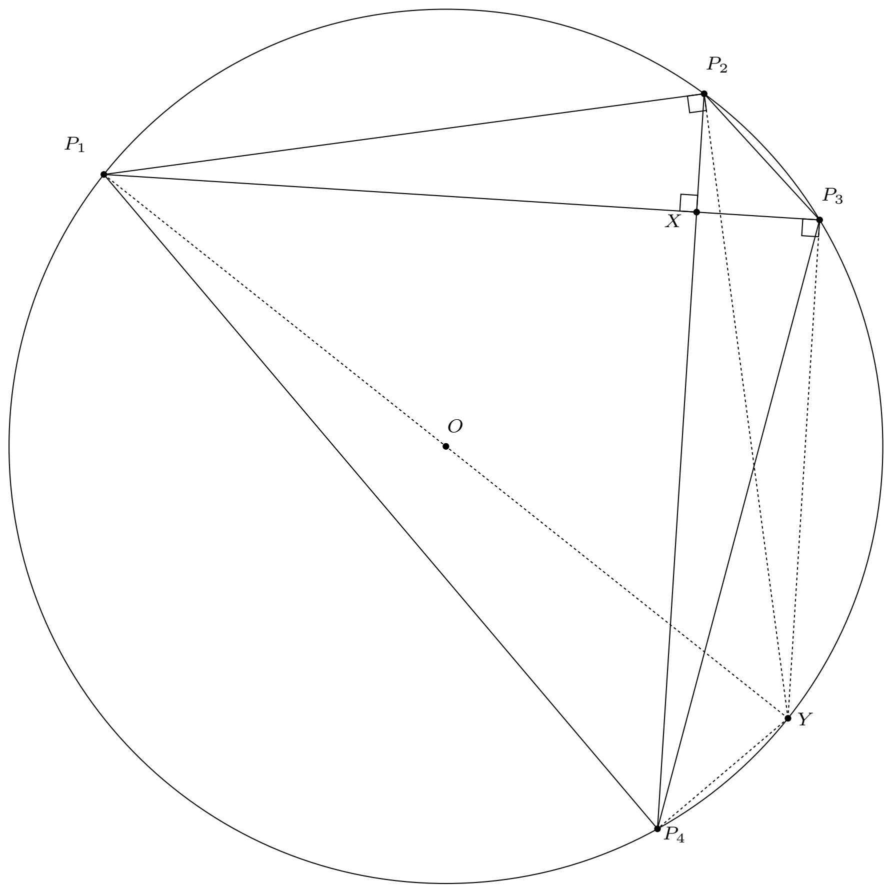

Figure 13: Problem 17

We observe that $\angle YP_2P_1 = \angle YP_3P_1 = 90°$, since both angles open up to a diameter. Since $\angle YP_3P_1 = \angle P_4XP_1$, we must have $\overline{YP_3} \parallel \overline{P_4P_2}$.

Since these two segments are parallel, $\angle P_2P_4P_3 = \angle YP_3P_4$. But since $YP_3P_2P_4$ is cyclic, we also have $\angle YP_3P_4 = \angle YP_2P_4$. Furthermore, $\angle YP_4P_3 = \angle YP_2P_3$ since $YP_3P_2P_4$ is cyclic. This means that:

$$\angle YP_4P_2 = \angle YP_4P_3 + \angle P_2P_4P_3 = \angle YP_2P_3 + \angle YP_2P_4 = \angle P_3P_2P_4$$

Since $\overline{YP_3} \parallel \overline{P_4P_2}$ and $\angle YP_4P_2 = \angle P_3P_2P_4$, quadrilateral $YP_3P_2P_4$ is an isosceles trapezoid. The diagonals are then equal in length:

$$YP_2 = P_3P_4$$

By the Pythagorean theorem on $\triangle YP_2P_1$ and the above equality:

$$D^2 = YP_2^2 + P_1P_2^2 = P_3P_4^2 + P_1P_2^2$$

By the Pythagorean theorem on $\triangle P_1XP_2$ and $\triangle P_3XP_4$, we find $P_1P_2^2 = XP_1^2 + XP_2^2$ and $P_3P_4^2 = XP_3^2 + XP_4^2$, respectively. Hence the equation above may be written as:

$$D^2 = XP_1^2 + XP_2^2 + XP_3^2 + XP_4^2$$

As desired. □

18. [**iTest 2008**] Two perpendicular planes intersect a sphere in two circles. These circles intersect in two points, A and B, such that $AB = 42$. If the radii of the two circles are 54 and 66, find R^2, where R is the radius of the sphere.

Solution: Suppose circle Γ has radius 54 with center G and circle Λ has radius 66 with center L. Let the foot of the altitude from G to $\overline{AB}$ be H (the midpoint of $\overline{AB}$). Let the center of the sphere be O. Then, by the Pythagorean theorem on $\triangle GHA$:

$$21^2 + GH^2 = 54^2 \Rightarrow GH^2 = 2475$$

By the Pythagorean theorem on $\triangle LHA$:

$$21^2 + LH^2 = 66^2 \Rightarrow LH^2 = 3915$$

By the Pythagorean theorem on $\triangle OGH$:

$$OH^2 = 2475 + 3915 = 6390$$

By the Pythagorean theorem on $\triangle OHA$:

$$R^2 = 21^2 + 6390 = \boxed{6831}$$

19. AIME [2008] In trapezoid $ABCD$ with $\overline{BC} \parallel \overline{AD}$, let $BC = 1000$ and $AD = 2008$. Let $\angle A = 37°$, $\angle D = 53°$, and M and N be the midpoints of $\overline{BC}$ and $\overline{AD}$, respectively. Find the length MN.

Solution: Let the length from A to the foot of the altitude from B to $\overline{AD}$, P, be a and let the length from D to the foot of the altitude from C to $\overline{AD}$, Q, be d. Let the lengths of those altitudes be h. Let X be the altitude from M to $\overline{AD}$. Then:

$$a = \frac{h}{\tan 37°}$$

And:

$$d = \frac{h}{\tan 53°} = h \tan 37°$$

From these we obtain $ad = h^2$. Furthermore, $a + d = 2008 - 1000 = 1008$, hence:

$$\frac{h}{\tan 37°} + h \tan 37° = 1008$$

From this we obtain:

$$h = \frac{1008}{\tan 37° + \cot 37°} = \frac{1008 \tan 37°}{\sec^2 37°} = 1008 \sin 37° \cos 37° = 504 \sin 74°$$

Using the second expression for h above, we quickly find $a = 1008 \cos^2 37°$.

We have $XP = 500$ and $NP = 1004 - a$ so $XN = a - 504 = 504 - 1008\cos^2 37°$. By the Pythagorean theorem on $\triangle MXN$:

$$\begin{aligned} MN^2 &= \left(504 - 1008\cos^2 37°\right)^2 + 504^2 \sin^2 74° \\ &= 1008^2 \cos^4 37° - 1008^2 \cos^2 37° + 504^2 \sin^2 74° + 504^2 \\ &= 1008^2 \cos^2 37° \left(\cos^2 37° - 1\right) + 504^2 \sin^2 74° + 504^2 \\ &= -1008^2 \cos^2 37° \sin^2 37° + 504^2 \sin^2 74° + 504^2 \\ &= -504^2 \sin^2 74° + 504^2 \sin^2 74° + 504^2 \\ &= 504^2 \end{aligned}$$

Hence $\boxed{MN = 504}$.